Institutionen för matematik
KTH
Chaotic Dynamical Systems, Fall 2010

Homework assignment 1

The exercises are due on October 1, 2010

1. Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that f has a periodic point of period 3. Show that f has a periodic point of period 2.

2. Let $f(x) = x^2 + x$. Find all fixed points of f. Where do nonfixed points go under iteration by f?

3. Exercise 3.14.

4. Let $f(x) = x + \frac{1}{10}\sin^2(\pi x)(\mod 1)$ be a map from the circle to itself (see page 25 in the text book). Where do points go under iteration of f?

5. Let $t \in \Sigma_2$. Define the stable set of t, $W^s(t)$, as the set of sequences s such that $d(\sigma^i(t), \sigma^i(s)) \to 0$ as $i \to \infty$. Identify all sequences in $W^s(t)$.

6. In the book *Chaotic Dynamical Systems* Devaney defines chaos as follows: Let X be a metric space (to avoid degenerate cases, assume that X is not a finite set). A continuous map $f : X \to X$ is said to be chaotic if

   i) f is topologically transitive (for any pair of open sets $U, V \subset X$ there exists a $k \geq 0$ such that $f^k(U) \cap V \neq \emptyset$);
   ii) the periodic points of f are dense in X;
   iii) f has sensitive dependence on initial conditions (there is a $\delta > 0$ such that for every $x \in X$ and every neighborhood N of x there exists a point $y \in N$ and an integer $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$).

   In this exercise we will see that in fact *i*) and *ii*) implies *iii*), i.e., if $f : X \to X$ is transitive and has dense periodic points then f has sensitive dependence on initial conditions.

a) Prove that there is a number $\delta_0 > 0$ such that for all $x \in X$ there exists a periodic point $q \in X$ whose orbit $O(q)$ is of distance at least $\delta_0/2$ from x.

Let $\delta = \delta_0/8$. Take an arbitrary $x \in X$ and let N be a neighborhood of x. Let $U = N \cap B_\delta(x)$, where the ball $B_\delta(x) = \{z \in X : d(x, z) < \delta\}$. Since the periodic points are dense there exists a periodic point $p \in U$. Let n be the period of p. By a) there is a periodic point $q \in X$ whose orbit $O(q)$ is of distance at least 4δ from x. Let

$$V = \bigcap_{n=0}^{n} f^{-i}(B_\delta(f^i(q))).$$

b) Verify that V is open and non-empty.

c) Conclude that there exists $y \in U$ and an integer $k \geq 0$ such that $f^k(y) \in V$.

d) Let j be the integer part of $k/n + 1$. Use the above construction to show that

$$d(f^{nj}(p), f^{nj}(y)) > 2\delta,$$

and conclude that f has sensitive dependence on initial conditions.