Institutionen för matematik
KTH
Chaotic Dynamical Systems, Fall 2010

Homework assignment 3

The exercises are due on November 26, 2010

1. Consider the diffeomorphism f_λ of the plane given by
$$f_\lambda : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} e^x - \lambda \\ -\frac{\lambda}{2}\arctan y \end{pmatrix}$$
where λ is a parameter.
a) Find all fixed points and periodic points of period 2 for f_λ.
b) Classify each of these periodic points as sinks, sources or saddles.
c) If the point is a saddle, identify and sketch the stable and unstable manifolds.

2. Consider the map $S(x, y) = (2x + y \mod 1, x + y \mod 1)$ on the two-torus $\mathbb{T}^2$ (see Challenge 2).
a) Prove Step 1 in Challenge 2 on page 93, for our particular map S.
b) Prove that $(0, 0)$ is the only fixed point.
c) Show that the periodic points are dense in $\mathbb{T}^2$.
d) Show that the stable and unstable manifolds of $(0, 0)$ are dense in $\mathbb{T}^2$.

3. The Lozi Attractor. Consider the piecewise linear map of the plane given by
$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + y - A|x| \\ Bx \end{pmatrix}$$
where A and B are parameters. Assume that $0 < B < 1$, $A > B + 1$ and $2A + B < 4$.
a) Prove that L has two fixed points, one of which lies in the first quadrant. We call this point p.
b) Prove that the unstable set $W^u(p)$ contains a straight line which intersects the $x-$axis at a point q and the y-axis at $L^{-1}(q)$.
c) Let l denote the straight line segment in $W^u(p)$ connecting q and $L^{-1}(q)$. Sketch $L(l)$ and $L^2(l)$.

d) Construct the triangle T with vertices at $q, L(q)$ and $L^2(q)$. Prove that T is a trapping region for L (i.e., show that $L(T)$ lies in the interior of T).

e) Use a computer to plot the forward orbits of points in T. The result is a picture of the Lozi attractor.

4. Let p_1, p_2 and p_3 be hyperbolic saddles of a smooth diffeomorphism f of $\mathbb{R}^2$. Assume that $W^u(p_1)$ intersects $W^s(p_2)$ transversally at some point q_1, and that $W^u(p_2)$ intersects $W^s(p_3)$ transversally at some point q_2. Use the λ−lemma (see page 415) to prove the following: given any two (small) neighborhoods $U \ni p_1$ and $V \ni p_3$, there exists a point $x \in U$ and an $n \geq 0$ such that $f^n(x) \in V$.

   (This is an example of property of diffeomorphisms called *accessibility* which is a well-studied notion in dynamics.)