Institutionen för matematik
KTH
Chaotic Dynamical Systems, Fall 2010

Homework assignment 4

The exercises are due on December 10, 2010

1. Consider
$$\frac{dx}{dt} = x - y - x(x^2 + 5y^2)$$
$$\frac{dy}{dt} = x + y - y(x^2 + y^2).$$
One can show that the origin is the only critical point (you can use it without verification).
a) Rewrite the system in polar coordinates, using $r'r = xx' + yy'$ and $\theta' = (xy' - yx')/r^2$
b) Find a circle of radius $r_1 > 0$, centered at the origin, such that all trajectories have a radially outward component on it.
c) Find a circle of radius $r_2 > r_1$, centered at the origin, such that all trajectories have a radially inward component on it.
d) Prove that the system has a limit cycle somewhere in the trapping region $r_1 \leq r \leq r_2$.

2. Gradient systems. Suppose the system can be written
$$\mathbf{x}' = -\nabla V(\mathbf{x}),$$
for some continuously differentiable, single-valued scalar function $V(\mathbf{x})$. Such a system is called a gradient system with potential function V.
Prove that closed orbits are impossible in gradient systems.
Hint: assume that there exists a closed orbit $\mathbf{x}(t)$ $(0 \leq t \leq T)$, and investigate the change in V after one circuit, i.e., investigate the integral
$$\int_0^T \frac{d}{dt} V(\mathbf{x}(t)dt.$$

3. One example, which has played a central role in the development of nonlinear dynamics, is given by the *van der Pol equation*
$$x'' + \mu(x^2 - 1)x' + x = 0$$

where $\mu \geq 0$ is a parameter. Historically, this equation arose in connection with the nonlinear electrical circuits used in the first radios. The equation looks like a simple harmonic oscillator, but with a nonlinear damping term $\mu(x^2 - 1)x'$. This term acts like ordinary positive damping for $|x| > 1$, but like negative damping for $|x| < 1$. In other words, it causes large-amplitude oscillations to decay, but it pumps them back up if they become too small.

Investigate the van der Pol equation, for $\mu = 1$, by doing "Challenge 7" on pages 316 - 320.