Lecture 2

1.10 Cell attachments

Let X be a topological space and $\alpha : S^{n-1} \longrightarrow X$ be a map. Consider the space $X \coprod D^n$ with the disjoint union topology. Consider further the set $X \coprod B^n$ and a function $f : X \coprod D^n \longrightarrow X \coprod B^n$ given by:

$$f(x) = \begin{cases} x & \text{if } x \in X \coprod B^n \\ \alpha(x) & \text{if } x \in S^{n-1} \subset D^n \end{cases}$$

The topological space consisting of the set $X \coprod B^n$ together with the quotient topology given by f is denoted by $X \cup_\alpha D^n$. We say that $X \cup_\alpha D^n$ is constructed out of X by attaching n-dimensional cell D^n along $\alpha : S^{n-1} \longrightarrow X$.

1.10.1 Excercise. Let $\alpha : S^{n-1} \longrightarrow X$ be a map. Show that the composition of the quotient map $f : X \coprod D^n \longrightarrow X \cup_\alpha D^n$ and the inclusion $X \coprod B^n \subset X \coprod D^n$, induced by the identity $\mathrm{id} : X \longrightarrow X$ and the inclusion $B^n \subset D^n$, is a continuous bijection. Is it an isomorphism?

1.10.2 Proposition. *If X is Hausdorff, then so is $X \cup_\alpha D^n$, for any map $\alpha : S^{n-1} \to X$.*

Proof. Let y_1 and y_2 be two distinct points in $X \cup_\alpha D^n$. There are 3 cases. First y_1 and y_2 are in X. Since X is Hausdorff, then there are two disjoint open subsets $y_1 \in U_1 \subset X$ and $y_2 \in U_2 \subset X$. The subsets $\alpha^{-1}(U_1)$ and $\alpha^{-1}(U_2)$ are open in S^{n-1}. Define:

$$V_i := \{x \in D^n \mid |x| > 1/2 \text{ and } x/|x| \in \alpha^{-1}(U_i)\} \subset D^n$$

Note that V_1 and V_2 are open and disjoint subsets of D^n. Finally set $W_i := U_i \cup (V_i \setminus \alpha^{-1}(U_i) \subset X \coprod B^n$. Note that $f^{-1}(W_i) = U_i \cup V_i \subset X \coprod D^n$. Thus W_i is open in $X \cup_\alpha D^n$. The subsets W_1 and W_2 are also disjoint and contain respectively y_1 and y_2.

Let $y_1 \in X$ and $y_2 \in B^n$. Define $W_1 := X \cup \{x \in B^n \mid |x| > (1 + |y_2|)/2\}$ and $W_2 := \{x \in B^n \mid |x| < (1 + |y_2|)/2\}$. Note that $f^{-1}(W_1) = X \cup \{x \in D^n \mid |x| > (1 + |y_2|)/2\}$ and $f^{-1}(W_2) = \{x \in D^n \mid |x| < (1 + |y_2|)/2\}$. These are open subsets. The sets W_i are then also open. They are disjoint and contain respectively y_1 and y_2.

Let y_1 and y_2 be two distinct points in B^n. Since B^n is Hausdorff, there are open disjoint subsets $y_1 \in W_1 \subset B^n$ and $y_2 \in W_2 \subset B^n$. The subset W_i is also open in Y. $\square$

1.10.3 Proposition. *If X is compact, then so is $X \cup_\alpha D^n$, for any map $\alpha : S^{n-1} \to X$.*

Proof. By previous proposition $X \cup_\alpha D^n$ is Hausdorff. Since $X \coprod D^n$ is compact, according to Proposition 1.9.7.(2), the space $X \cup_\alpha D^n$, as the image of $f : X \coprod D^n \longrightarrow X \cup_\alpha D^n$, is also compact. $\square$

1.10.4 Example. Consider the one point space D^0. Let $\alpha : S^{n-1} \longrightarrow D^0$ be the unique map. The space $D^0 \cup_\alpha D^n$ is isomorphic to S^n. To construct the isomorphism consider a function $g : D^0 \coprod B^n \longrightarrow S^n$ defined as follows:

$$g(x) = \begin{cases} (0, \cdots, 0, -1) & \text{if } x \in D^0 \\ (2\sqrt{\frac{1-|x|}{|x|}}x, 1 - 2|x|) & \text{if } x \in B^n \setminus \{(0, \cdots, 0)\} \\ (0, \cdots, 0, 1) & \text{if } x = (0, \cdots, 0) \in B^n \end{cases}$$

We claim that g is a bijection and that the composition of the quotient function $f : D^0 \coprod D^n \longrightarrow D^0 \coprod B^n$ and g is a continuous function $gf : D^0 \coprod D^n \longrightarrow S^n$. To see this we need to show that the restriction of gf to the components D^0 and D^n are continuous. This is true for the first restriction since all functions out of D^0 are continuous. The other restriction is given by the formula:

$$D^n \ni x \mapsto \begin{cases} (2\sqrt{\frac{1-|x|}{|x|}}x, 1 - 2|x|) \in S^n & \text{if } x \in D^n \setminus \{(0, \cdots, 0)\} \\ (0, \cdots, 0, 1) \in S^n & \text{if } x = (0, \cdots, 0) \in D^n \end{cases}$$

whose continuity can be checked using 1.7.4. Thus the function g defines a continuous map, denoted by the same symbol, $g : D^0 \cup_\alpha D^n \longrightarrow S^n$. As both spaces $D^0 \cup_\alpha D^n$ and S^n are compact, this map g must be then an isomorphism.

1.11 Real projective spaces

Consider the Euclidean space $\mathbf{R}^{n+1}$ ($n \geq 0$). The symbol $\mathbf{RP}^n$ denotes the set of 1-dimensional $\mathbf{R}$-vector subspaces of $\mathbf{R}^{n+1}$. Such subspaces are also called lines in $\mathbf{R}^{n+1}$. For example since $\mathbf{R}$ is 1-dimensional $\mathbf{R}$-vector space, it has only one 1-dimensional $\mathbf{R}$-vector subspace, and hence $\mathbf{RP}^0$ is just a point.

Define $\pi : S^n \longrightarrow \mathbf{RP}^n$ to be the function that assigns to a vector $v \in S^n \subset \mathbf{R}^{n+1}$ the $\mathbf{R}$-linear subspace generated by v. Explicitly $\pi(v) := \{rv \mid r \in \mathbf{R}\}$.

1.11.1 Definition. *The set* $\mathbf{RP}^n$ *together with the quotient topology induced by* $\pi : S^n \longrightarrow \mathbf{RP}^n$ *is called the* n*-dimensional real projective space.*

In the rest of this section we will identify the projective spaces as spaces build by attaching cells. For that we need:

1.11.2 Proposition. $\mathbf{RP}^n$ *is a compact space.*

Proof. Since S^n is compact, according to Proposition 1.9.7.(2), to show that $\mathbf{RP}^n$ is compact it is enough to prove that it is Hausdorff. Let L_1 and L_2 be two distinct points in $\mathbf{RP}^n$, i.e., two distinct 1-dimensional $\mathbf{R}$-linear subspaces in $\mathbf{R}^{n+1}$. Let v_1 and v_2 be two points in S^n which generate the lines L_1 and L_2 respectively. Let $r = \min\{|v_1 - v_2|, |v_1 + v_2|\}$. Define U_1 to be the subset of $\mathbf{PR}^n$ of all the lines which are generated by vectors v such that $\min\{|v_1 - v|, |v_1 + v|\} < r/2$. Define U_2 to be subset of $\mathbf{RP}^n$ of all the lines which are generated by vectors v such that $\min\{|v_2 - v|, |v_2 + v|\} < r/2$. It is clear that the subsets U_1 and U_2 are disjoint and $L_1 \in U_1$ and $L_2 \in U_2$. We claim that these sets are also open. Note that:

$$\pi^{-1}(U_i) = \{w \in S^n \mid |v_i - w| < r/2\} \cup \{w \in S^n \mid |v_i + w| < r/2\}$$

Since it is an open subset in S^n, U_i is open in $\mathbf{RP}^n$. $\square$

We can use the map $\pi : S^n \longrightarrow \mathbf{RP}^n$ to attach a cell and build a new topological space $\mathbf{RP}^n \cup_\pi D^{n+1}$.

1.11.3 Proposition. *The space* $\mathbf{RP}^n \cup_\pi D^{n+1}$ *is isomorphic to* $\mathbf{RP}^{n+1}$.

Proof. Since both spaces $\mathbf{RP}^n \cup_\pi D^{n+1}$ and $\mathbf{RP}^{n+1}$ are compact, to show that they are isomorphic we need to construct a continuous bijection between them. Let us denote by $e : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^{n+2}$, respectively $e : S^n \longrightarrow S^{n+1}$, the functions which assigns to an element $x \in \mathbf{R}^{n+1}$ the element $(x, 0) \in \mathbf{R}^{n+2}$. Note that both of these functions are continuous. Define $i : \mathbf{RP}^n \longrightarrow \mathbf{RP}^{n+1}$ to be a function that assigns to a line $L \subset \mathbf{R}^{n+1}$, the line in $e(L) \subset \mathbf{R}^{n+2}$. Note that there is a commutative diagram:

$$\begin{tikzcd} S^n \arrow[r, "e"] \arrow[d, "\pi"'] & S^{n+1} \arrow[d, "\pi"] \\ \mathbf{RP}^n \arrow[r, "i"] & \mathbf{RP}^{n+1} \end{tikzcd}$$

Since the composition $i\pi = \pi e$ is continuous, then i is also continuous. The map $i : \mathbf{RP}^n \longrightarrow \mathbf{RP}^{n+1}$ is called the standard inclusion.

Define a function $g : \mathbf{RP}^n \coprod B^{n+1} \to \mathbf{RP}^{n+1}$ as follows:

$$g(x) = \begin{cases} i(x) & \text{if } x \in \mathbf{RP}^n \\ \text{the line generated by } (x, \sqrt{1-|x|^2}) \in \mathbf{R}^{n+2} & \text{if } x \in B^{n+1} \end{cases}$$

Note that the composition of the function g with the quotient map $f : \mathbf{RP}^n \coprod D^{n+1} \longrightarrow \mathbf{RP}^n \cup_\pi D^{n+1}$ is continuous. Thus g defines a continuous map $g : \mathbf{RP}^n \cup_\pi D^{n+1} \longrightarrow \mathbf{RP}^{n+1}$. Note that this map is a bijection. Since the spaces are compact we can conclude that this map is an isomorphism. □

1.11.4 Example. Let $n = 0$. In this case $\mathbf{RP}^0$ is just a point and as a topological space it is isomorphic to D^0. Thus there is only one map $\pi : S^0 \longrightarrow \mathbf{RP}^0$. The space $\mathbf{RP}^1$ is then isomorphic to $D^0 \cup_\pi D^1$, which by Example 1.10.4, is isomorphic to S^1. It follows then that $\mathbf{RP}^1$ is isomorphic to S^1. We would like to identify the map $\pi : S^1 \longrightarrow \mathbf{RP}^1 = S^1$. Note that this map sands the elements x and $-x$ to the same point. If we think about S^1 as a subset of the complex numbers $\mathbf{C}$ of length 1, then the multiplication map $S^1 \ni z \mapsto z^2 \in S^1$ also sends x and $-x$ to the same point. One can then check directly that the map $\pi : S^1 \longrightarrow \mathbf{RP}^1 = S^1$ is given by this multiplication map.

1.12 Complex projective spaces

Consider the complex vector space $\mathbf{C}^{n+1}$ ($n \geq 0$). The symbol $\mathbf{CP}^n$ denotes the set of 1-dimensional $\mathbf{C}$-vector subspaces of $\mathbf{C}^{n+1}$. Such subspaces are also called complex lines in $\mathbf{C}^{n+1}$. For example since $\mathbf{C}$ is 1-dimensional $\mathbf{C}$-vector space, it has only one 1-dimensional $\mathbf{C}$-vector subspace, and hence $\mathbf{CP}^0$ is just a point.

Note that the subspace $\{z \in \mathbf{C}^{n+1} \mid |z| = 1\}$ is isomorphic to S^{2n+1}. Define $\pi : S^{2n+1} \longrightarrow \mathbf{CP}^n$ to be the function that assigns to a vector z the 1-dimensional $\mathbf{C}$ vector subspace of $\mathbf{C}^{n+1}$ generated by z. Explicitly $\pi(z) = \{sz \mid s \in \mathbf{C}\}$.

1.12.1 Definition. *The set $\mathbf{CP}^n$, together with the quotient topology induced by $\pi : S^{2n+1} \longrightarrow \mathbf{CP}^n$, is called the n-dimensional complex projective space.*

In the rest of this section we will identify the complex projective spaces as spaces build by attaching cells.

1.12.2 Proposition. *(1) $\mathbf{CP}^n$ is a compact space.*

(2) The space $\mathbf{CP}^n \cup_\pi D^{2n+2}$ *is isomorphic to* $\mathbf{CP}^{n+1}$.

Proof. (1): Since S^{2n+1} is compact, according to Proposition 1.9.7.(2), to show that $\mathbf{CP}^n$ is compact it is enough to prove that it is Hausdorff. Let L_1 and L_2 be two distinct points in $\mathbf{CP}^n$, i.e., two distinct 1-dimensional $\mathbf{C}$-linear subspaces in $\mathbf{C}^{n+1}$. Let z and y be two points in S^{2n+1} which generate the lines L_1 and L_2 respectively. Let $R = \min\{|sz - ty| \mid s, t \in S^1 \subset \mathbf{C}\}$. Note that because S^1 is compact, $R > 0$. Define V and U to be the following open subsets of S^{2n+1}:

$$V := \{x \in S^{2n+1} \mid |x - z| < R/2\} \quad U := \{x \in S^{2n+1} \mid |x - y| < R/2\}$$

Since for any $s \in S^1 \subset \mathbf{C}$, $sV = V$ and $sU = U$, we have equalities:

$$\pi^{-1}\pi(V) = V \qquad \pi^{-1}\pi(U) = U$$

Thus the subsets $\pi(V)$ and $\pi(U)$ are open in $\mathbf{CP}^n$. As they do not intersect and contain respectively L_1 and L_2, we can conclude that $\mathbf{CP}^n$ is Hausdorff.

(2): Since both spaces $\mathbf{CP}^n \cup_\pi D^{2n+2}$ and $\mathbf{CP}^{n+1}$ are compact, to show that they are isomorphic we need to construct a continuous bijection between them. Let us denote by $e : \mathbf{C}^{n+1} \longrightarrow \mathbf{C}^{n+2}$, respectively $e : S^{2n+1} \longrightarrow S^{2n+3}$, the functions which assigns to an element $z \in \mathbf{C}^{n+1}$ the element $(z, 0) \in \mathbf{C}^{n+2}$. Note that both of these functions are continuous. Define $i : \mathbf{CP}^n \longrightarrow \mathbf{CP}^{n+1}$ to be a function that assigns to a line $L \subset \mathbf{C}^{n+1}$, the line in $e(L) \subset \mathbf{C}^{n+2}$. Note that there is a commutative diagram:

$$\begin{tikzcd} S^{2n+1} \arrow[r, "e"] \arrow[d, "\pi"'] & S^{2n+3} \arrow[d, "\pi"] \\ \mathbf{CP}^n \arrow[r, "i"] & \mathbf{CP}^{n+1} \end{tikzcd}$$

Since the composition $i\pi = \pi e$ is continuous, then i is also continuous. The map $i : \mathbf{CP}^n \longrightarrow \mathbf{CP}^{n+1}$ is called the standard inclusion.

Define a function $g : \mathbf{CP}^n \coprod B^{2n+2} \to \mathbf{CP}^{n+1}$ as follows:

$$g(z) = \begin{cases} i(z) & \text{if } z \in \mathbf{CP}^n \\ \text{the line generated by } (z,1) \in \mathbf{C}^{n+2} & \text{if } z \in B^{2n+2} \subset \mathbf{C}^{n+1} \end{cases}$$

Note that the composition of the function g with the quotient map $f : \mathbf{CP}^n \coprod D^{2n+2} \longrightarrow \mathbf{CP}^n \cup_\pi D^{2n+2}$ is continuous. Thus g defines a continuous map $g : \mathbf{CP}^n \cup_\pi D^{2n+2} \longrightarrow \mathbf{CP}^{n+1}$. Note that this map is a bijection. Since the spaces are compact we can conclude that this map is an isomorphism. □

1.12.3 Excercise. Let $1 \leq i \leq n+1$. Define $U_i \subset \mathbf{CP}^n$ to be the image of $\{z \in \mathbf{S}^{2n+1} \mid z_i \neq 0\}$ via π. Show that U_i is an open subset of $\mathbf{CP}^n$. Show further that the function that assigns to an element $z \in \mathbf{C}^n$, the line in $\mathbf{C}^{n+1}$ generated by $(z_1, \cdots, z_{i-1}, 1, z_i, \cdots z_n)$, is an isomorphism between $\mathbf{C}^n$ and U_i.

1.12.4 Example. Let $n = 0$. In this case $\mathbf{CP}^0$ is just a point and as a topological space it is isomorphic to D^0. Thus there is only one map $\pi : S^1 \longrightarrow \mathbf{CP}^0$. The space $\mathbf{CP}^1$ is then isomorphic to $D^0 \cup_\pi D^2$, which by Example 1.10.4, is isomorphic to S^2. It follows then that $\mathbf{CP}^1$ is isomorphic to S^2.

1.13 Tautological line bundles

Let us choose $n \geq 0$. Consider the following subspace of the product $\mathbf{CP}^n \times \mathbf{C}^{n+1}$:

$$E := \{(L, z) \in \mathbf{CP}^n \times \mathbf{C}^{n+1} \mid z \in L\}$$

1.13.1 Definition. *The composition $E \subset \mathbf{CP}^n \times \mathbf{C}^{n+1} \xrightarrow{\mathrm{pr}_{\mathbf{CP}^n}} \mathbf{CP}^n$ is denoted by λ_n and called the tautological line bundle.*

Note that for any point $L \in \mathbf{CP}^n$, the preimage $(\lambda_n)^{-1}(L)$ is can be identified with the line $L \subset \mathbf{C}^{n+1}$. In this way $(\lambda_n)^{-1}(L)$ becomes a 1-dimensional $\mathbf{C}$ vector space. Furthermore, consider the open subset $U_i \subset \mathbf{CP}^n$ (see 1.12.3) and the following maps:

$$\begin{tikzcd}
(\lambda_n)^{-1}(U_i) \arrow[r, hook] \arrow[d, hook] \arrow[rr, bend left, "\phi_i"] & U_i \times \mathbf{C}^{n+1} \arrow[r, two heads, "\mathrm{id}\times\mathrm{pr}_i"] \arrow[d, hook] & U_i \times \mathbf{C} \arrow[d, hook] \\
E \arrow[r, hook] & \mathbf{CP}^n \times \mathbf{C}^{n+1} \arrow[r, two heads, "\mathrm{id}\times\mathrm{pr}_i"] & \mathbf{CP}^n \times \mathbf{C}
\end{tikzcd}$$

1.13.2 Proposition. *For any i, the map $\phi_i : (\lambda_n)^{-1}(U_i) \longrightarrow U_i \times \mathbf{C}$ is an isomorphism.*

1.13.3 Excercise. Prove the above proposition.

1.14 Homotopy relation

Recall that I denotes the unit interval $[0, 1] \subset \mathbf{R}$. We are going to use it to define a relation on continuous maps with the same domain and range:

1.14.1 Definition. *A map $f : X \longrightarrow Y$ is homotopic to $g : X \longrightarrow Y$ if there is a map $S : X \times I \longrightarrow Y$ such that, for any $x \in X$, $S(x, 0) = f(x)$ and $S(x, 1) = g(x)$. Any such map S is called a homotopy between f and g.*

Homotopy is an equivalence relation on the set of continuous maps between X and Y and is preserved by compositions:

1.14.2 Proposition. *(1) If $f : X \longrightarrow Y$ is homotopic to $g : X \longrightarrow Y$ then g is homotopic to f (symmetry of the homotopy relation).*

(2) If $f : X \longrightarrow Y$ is homotopic to $g : X \longrightarrow Y$ and $g : X \longrightarrow Y$ is homotopic to $h : X \longrightarrow Y$, then f is homotopic to h (transitivity of the homotopy relation).

(3) If $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ are homotopic, then, for any $h : Y \longrightarrow Z$, so are the compositions $hf : X \longrightarrow Z$ and $hg : X \longrightarrow Z$.

(4) If $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ are homotopic, then, for any $h : Z \longrightarrow X$, so are the compositions $fh : Z \longrightarrow Y$ and $gh : Z \longrightarrow Y$.

Proof. (1): Let $S : X \times I \longrightarrow Y$ be a homotopy between f and g. Define $S' : X \times I \longrightarrow Y$ as $S'(x, t) = S(x, 1 - t)$. Note that S' is a homotopy between g and f.
(2): Let $S : X \times I \longrightarrow Y$ be a homotopy between f and g and $S' : X \times I \longrightarrow Y$ be a homotopy between g and h. Define $S'' : X \times I \longrightarrow Y$ by the formula:

$$S''(x, t) = \begin{cases} S(x, 2t) & \text{if } 0 \leq t \leq 1/2 \\ S'(x, 2t - 1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

Note that $S''(x, 0) = f(x)$ and $S''(x, 1) = h(x)$. Thus, if continuous, S'' would be a homotopy between f and h. To see that S'' is continuous consider the following compositions:

$$X \times [0, 1/2] \xrightarrow{\alpha} X \times I \xrightarrow{S} Y$$

$$X \times [1/2, 1] \xrightarrow{\beta} X \times I \xrightarrow{S'} Y$$

where $\alpha(x, t) = (x, 2t)$ and $\beta(x, t) = (x, 2t-1)$. These composition are clearly continuous. It follows that, if D is closed in Y, then so are $(S\alpha)^{-1}(D) \subset X \times [0, 1/2] \subset X \times I$ and $(S'\beta)^{-1}(D) \subset X \times [1/2, 1] \subset X \times I$. The sum $(S\alpha)^{-1}(D) \cup (S'\beta)^{-1}(D)$ is then also closed in $X \times I$. Note however that this sum coincide with $(S'')^{-1}(D)$. The function S'' is therefore continuous.
(3): If $S : X \times I \longrightarrow Y$ is a homotopy between f and g, then hS is a homotopy between hf and hg.

(4): Let $S : X \times I \longrightarrow Y$ be a homotopy between f and g. Define $S' : Z \times I \longrightarrow Y$ by the formula $S'(z,t) = S(h(x),t)$. This is a homotopy between fh and gh. □

The fundamental example of homotopic maps are the inclusions $\mathrm{in}_0 : X \longrightarrow X \times I$ and $\mathrm{in}_1 : X \longrightarrow X \times I$, where $\mathrm{in}_0(x) = (x,0)$ and $\mathrm{in}_1(x) = (x,1)$.

Homotopy relation can be used to define:

1.14.3 Definition. *A map $f : X \longrightarrow Y$ is called a homotopy equivalence if there is a map $g : Y \longrightarrow X$ such that the compositions fg and gf are homotopic respectively to the identity maps $id : Y \longrightarrow Y$ and $id : X \longrightarrow X$.*

Two spaces X and Y are said to be homotopy equivalent if there is a homotopy equivalence $f : X \longrightarrow Y$.

A space is called contractible if it is homotopy equivalent to the one point space D^0.

The homotopy equivalence relation on spaces is an equivalence relation. It is a weaker relation than an isomorphism. Two isomorphic spaces are clearly homotopy equivalent.

1.14.4 Proposition. *(1) If $D \subset \mathbf{R}^n$ is non-empty and convex (the interval between any two points in D is subset of D), then D is contractible.*

(2) The spaces $\mathbf{R}^n$, D^n, Δ^n, and B^n are contractible.

Proof. (1): Let us choose a point $x \in D$. Define $f : D^0 \longrightarrow D$ to be given by $f(0) = x$ and $g : D \longrightarrow D^0$ to be the unique map. Clearly $gf = \mathrm{id}$. We need to show that the composition $fg : D \longrightarrow D$ is homotopic to $\mathrm{id} : D \longrightarrow D$. Define $S : D \times I \longrightarrow D$ by the formula: $S(y,t) = tx + (1-t)y$. It is well define since D is convex. Note that $s(y,0) = y$ and $s(y,1) = fg$. Thus S is a homotopy between id and fg.

(2): This follows from statement (1) as all these spaces are convex. □

1.14.5 Proposition. *(1) Let $D \subset \mathbf{R}^n$ be convex and $x \in D$ a point for which there is $r > 0$ such that $B(x,r) \subset D$. Then the space $D \setminus \{x\}$ is homotopy equivalent to S^{n-1}.*

(2) Let $n > 0$. The spaces $\mathbf{R}^n \setminus \{0\}$, $D^n \setminus \{0\}$, $\Delta^n \setminus \{\big(1/(n+1), \ldots, 1/(n+1)\big)\}$, and $B^n \setminus \{0\}$ are homotopy equivalent to S^{n-1}.

Proof. Since D is convex, the space $S(x,r/2)$ is a subspace of D. This space $S(x,r/2)$ is isomorphic to S^{n-1}. Thus to show the statement it is enough to show that $D \setminus \{x\}$ is homotopy equivalent to $S(x,r/2)$. Set a map $f : S(x,r/2) \longrightarrow D \setminus \{x\}$ to be the inclusion and $g : D \setminus \{x\} \longrightarrow S(x,r/2)$

to be defined by the formula $g(y) = x + \frac{r(y-x)}{2|y-x|}$. It is straight forward to check that gf is $\mathrm{id} : S(x, r/2) \longrightarrow S(x, r/2)$. We need to show that fg is homotopic to $\mathrm{id} : D \setminus \{x\} \longrightarrow D \setminus \{x\}$. Define $H : (D \setminus \{x\}) \times I \longrightarrow D \setminus \{x\}$:

$$H(y, t) := ty + (1-t)\big(x + \frac{r(y-x)}{2|y-x|}\big)$$

Note that $H(y, 1) = \mathrm{id}$ and $H(y, 0) = fg(y)$. □

1.15 $\pi_0(X)$

Let X be a topological space. Note that maps $f : D^0 \longrightarrow X$ can be identified with elements of X. Such a map is determined by where it sends the only point of D^0. In this way we can think about elements of X as maps from the one point space D^0 to X. The homotopy relation on such maps can be then rephrased in terms of elements of X as follows: two points $x_0, x_1 \in X$ are homotopic if there is a map $\alpha : I \longrightarrow X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Such continuous maps are called paths between x_0 and x_1. To have a language to describe this particular situation we are going to use the following definition:

1.15.1 Definition. *Two points $x_0 \in X$ and $x_1 \in X$ are said to be in the same path component if there is a path $\alpha : I \longrightarrow X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$.*

Since homotopy relation on maps is an equivalence relation (see Proposition 1.14.2), we get that being in the same path component is also an equivalence relation on the set of elements of X. We can then consider the equivalence classes of this relation and define:

1.15.2 Definition. *The set of equivalence classes of the relation "being in the same path component" on the set of points of X is denoted by $\pi_0(X)$.*

A space is called path connected if $\pi_0(X)$ consists of one point, i.e., if all pairs of points in X are in the same path component.

1.15.3 Excercise. Show that, for any $n \geq 0$, Δ^n is path connected.

We are going to denote elements in $\pi_0(X)$ by $[x]$, where $x \in X$ is a point in the given equivalence class. For such an element $[x] \in \pi_0(X)$, let us denote by $X_{[x]}$ the subspace of X consisting of all points in X that are in the same path component as x. These subspaces of X are called path connected components of X. Note that these subspaces are path connected, i.e., $\pi_0(X_{[x]})$ is the one point set.

1.15.4 Excercise. Show that if Y is path connected, then for any map $f : Y \longrightarrow X$, there is $x \in X$ such that $f(Y) \subset X_{[x]}$.