Lecture 4

1.19 Complex vector bundles

In this section we are going to define complex vector bundles. We start with discussing the simplest type of vector bundles, the product bundles. Let U be a topological space and $f : U \times \mathbf{C}^n \longrightarrow U \times \mathbf{C}^m$ be a function (I do not assume that f is continuous) such that:

(1) $\mathrm{pr}_U f = \mathrm{pr}_U$, i.e., the following diagram commutes:

$$\begin{array}{ccc} U \times \mathbf{C}^n & \xrightarrow{\ f\ } & U \times \mathbf{C}^m \\ & {\scriptstyle \mathrm{pr}_U}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_U} & \\ & U & \end{array}$$

(2) the induced map $f(x, -) : \{x\} \times \mathbf{C}^n \longrightarrow \{x\} \times \mathbf{C}^m$ is linear for all $x \in U$. We will often denote this map by $f_x : \mathbf{C}^n \longrightarrow \mathbf{C}^m$.

Such a function f induces then a function $\hat{f} : U \longrightarrow \hom(\mathbf{C}^n, \mathbf{C}^m)$ which maps $x \in U$ to the linear function $f_x \in \hom(\mathbf{C}^n, \mathbf{C}^m)$.

1.19.1 Lemma. *f is continuous if and only if $\hat{f}$ is continuous.*

Proof. We will show only one implication. The other is left as an exercise. Assume that $\hat{f}$ is continuous. Consider the following composition:

$$\begin{array}{ccccc} (x, v) & \longmapsto & (x, \hat{f}(x), v) & & \\ U \times \mathbf{C}^n & \longrightarrow & U \times \hom(\mathbf{C}^n, \mathbf{C}^m) \times \mathbf{C}^n & \longrightarrow & U \times \mathbf{C}^n \\ & & (x, \phi, v) & \longmapsto & (x, \phi(v)) \end{array}$$

Note that both of the above functions are continuous, and hence so is their composition. Note finally that this composition maps (x, v) to $(x, \hat{f}(x)(v)) = f(x, v)$. We can conclude that f is then continuous. $\square$

1.19.2 Excercise. Finish the proof of the above proposition. It remains to show the implication: if f is continuous, then so is $\hat{f}$.

1.19.3 Definition. *(1) A complex vector bundle is a map* $p : E \longrightarrow X$*; together with a structure of* $\mathbf{C}$*-vector space on* $p^{-1}(x)$ *for any* $x \in X$*; The map* p*, and the vector spaces* $p^{-1}(x)$*, for* $x \in X$*, are required to satisfy the following condition: for any* $x \in X$*, there is an open set* $x \in U \subset X$ *and an isomorphism* $\phi : U \times \mathbf{C}^n \longrightarrow p^{-1}(U)$ *such that:*

- $p\phi = pr_U$*, i.e., the following diagram commutes:*

$$\begin{array}{ccc} U \times \mathbf{C}^n & \xrightarrow{\ \phi\ } & p^{-1}(U) \\ & {\scriptstyle pr_U} \searrow \quad \swarrow {\scriptstyle p} & \\ & U & \end{array}$$

- *for any* $y \in U$*, the induced isomorphism* $\phi(y, -) : \mathbf{C}^n \longrightarrow p^{-1}(y)$ *is* $\mathbf{C}$*-linear.*

(2) Let $p : E \longrightarrow X$ *be a complex vector bundle. We say that* p *is trivial on an open subset* $U \subset X$ *if there is an isomorphism* $\phi : U \times \mathbf{C}^n \longrightarrow p^{-1}(U)$ *such that:*

- $p\phi = pr_U$*, i.e., the following diagram commutes:*

$$\begin{array}{ccc} U \times \mathbf{C}^n & \xrightarrow{\ \phi\ } & p^{-1}(U) \\ & {\scriptstyle pr_U} \searrow \quad \swarrow {\scriptstyle p} & \\ & U & \end{array}$$

- *for any* $y \in U$*, the induced isomorphism* $\phi(y, -) : \mathbf{C}^n \longrightarrow p^{-1}(y)$ *is* $\mathbf{C}$*-linear.*

Any such isomorphism ϕ *is called a trivialization of* p *on* U*.*

(3) Let $p : E \longrightarrow X$ *and* $q : F \longrightarrow Y$ *be complex vector bundles. A vector bundle map between* p *and* q *is a pair of continuous maps* $f_0 : X \longrightarrow Y$ *and* $f : E \longrightarrow F$ *such that:*

- $f_0 p = qf$*, i.s., the following diagram commutes:*

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\ f_0\ } & Y \end{array}$$

- *For any $x \in X$, the induced map $f : p^{-1}(x) \longrightarrow q^{-1}(f(x))$ is linear.*

1.19.4 Example. Let $X = \mathbf{C}P^n$, $E = \{(L, z) \in \mathbf{C}P^n \times \mathbf{C}^{n+1} \mid z \in L\}$, and $\lambda_n : E \longrightarrow X$ is the function that maps (L, z) to L. Thus λ is the composition of the inclusion $E \subset \mathbf{C}P^n \times \mathbf{C}^{n+1}$ and the projection $\mathbf{C}P^n \times \mathbf{C}^{n+1} \longrightarrow \mathbf{C}P^n$. It is then a continuous function. Note that for any $L \in \mathbf{C}P^n$, $\lambda_n^{-1}(L) = L \subset \mathbf{C}^{n+1}$ is a 1-dimensional complex vector subspace of $\mathbf{C}^{n+1}$. We claim that with this choice of complex vector structures on $\lambda_n^{-1}(L)$, for $L \in \mathbf{C}P^n$, λ is a vector bundle.

Let $U_i \subset \mathbf{C}P^n$ be the set of these lines that contain a vector of the form $(z_1, \ldots, z_{n+1})$ with $z_i = 1$ (see 1.12.3). Let $\phi : U_i \times \mathbf{C} \longrightarrow \lambda^{-1}(U_i)$ be a function that maps $(L = \text{line generated by}(z_1, \ldots, z_i = 1, \ldots, z_{n+1}), r)$ to $(L, r(z_1, \ldots, z_{n+1})$. Then ϕ is a continuous isomorphism.

1.20 Maps of vector bundles

Let $p : E \longrightarrow X$ and $q : F \longrightarrow Y$ be complex vector bundles and $f_0 : X \longrightarrow Y$ be a continuous map. Assume that $f : E \longrightarrow F$ is function (not necessarily continuous) such that $f_0 p = qf$, i,e., the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f_0} & Y \end{array}$$

Assume that $\{U_i\}_{i \in I}$ is an open cover of X such that p is trivial over U_i for any i. Let $\phi_i : U_i \times \mathbf{C}^n \longrightarrow p^{-1}(U_i)$ be a trivialization of p on U_i. Let that $\{V_j\}_{j \in J}$ be an open cover of Y such that q is trivial over V_j for any j. Let $\psi_j : V_i \times \mathbf{C}^m \longrightarrow q^{-1}(V_i)$ be a trivialization of q on V_j. Consider an open covering $\{W_{i,j} := U_i \cap f_0^{-1}(V_j)\}_{i \in I, j \in J}$ of X. All these functions fit into the following commutative diagram:

$$\begin{array}{ccccc} W_{i,j} \times \mathbf{C}^n & & \xrightarrow{\quad \psi_j^{-1} f \phi_i \quad} & & V_j \times \mathbf{C}^n \\ & \searrow^{\phi_i} & & \nearrow^{\psi_j^{-1}} & \\ & p^{-1}(W_{i,j}) & \xrightarrow{f} & q^{-1}(V_j) & \\ & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} & \\ & W_{i,j} & \xrightarrow{f_0} & V_j & \end{array}$$

1.20.1 Excercise. Show that (f_0, f) is a vector bundle map between p and q if and only if the following two conditions are satisfied:

- For any $x \in W_{i,j}$, $\psi_j^{-1} f \phi_i(x, -) : \mathbf{C}^n \longrightarrow \mathbf{C}^m$ is linear,
- the induced function $\widehat{\psi_j^{-1} f \phi_i} : W_{i,j} \longrightarrow \hom(\mathbf{C}^n, \mathbf{C}^m)$ is continuous (see 1.19.1).

This can be used to construct maps of vector bundles:

1.20.2 Corollary. *Let $p : E \longrightarrow X$ and $q : F \longrightarrow Y$ be complex vector bundles and $(f_0 : X \longrightarrow Y, f : E \longrightarrow F)$ be a vector bundle map between p and q. Then (f_0, f) is an isomorphism if and only if f_0 is an isomorphism and, for any $x \in X$, the induced map $f : p^{-1}(x) \longrightarrow q^{-1}(f_0(x))$ is a linear isomorphism.*

Proof. If (f_0, f) is an isomorphism then it has an inverse and hence the required conditions are necessary.

Assume that the above conditions are satisfied. The inverse f_0^{-1} is continuos by definition. Continuity of the inverse f^{-1} follows from the above exercise and the fact that the inverse function $\mathrm{GL}(\mathbf{C}^n) \ni \alpha \mapsto \alpha^{-1} \in \mathrm{GL}(\mathbf{C}^n)$ is continuous. □