KTH Teknikvetenskap

SF2729 Groups and Rings
Final Exam
Friday, May 27, 2011

Time: 14.00-18.00
Allowed aids: none
Examiner: Mats Boij

This final exam consists of two parts; Part I (groups part) and Part II (rings part). The final credit for Part I will be based on the maximum of the results on the midterm exam and Part I in the final exam.

Each problem can give up to 6 points. In the first problem of each part, you are guaranteed a minimum given by the result of the corresponding homework assignment. If you have at least 2 points from HW1, you cannot get anything from Part a) of Problem 1 of Part I, if you have at least 4 points from HW1 you cannot get anything from Part a) or Part b) of Problem 1 of Part I. Similarly for HW2 and Problem 1 of Part II.

The minimum requirements for the various grades are according to the following table:

Grade	**A**	**B**	**C**	**D**	**E**
Total credit	30	27	24	21	18
From Part I	13	12	11	9	8
From Part II	13	12	11	9	8

Present your solutions to the problems in a way such that arguments and calculations are easy to follow. Provide detailed arguments to your answers. An answer without explanation will give no points.

Part I - Groups

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(1) (a) The axioms of a group only state the existence of an identity element e such that $a * e = e * a = a$ for all a in the group. Show that this element is unique. **(2)**

(b) The dihedral group D_{2n} can be defined as the symmetries of a regular n-gon. Show that the center of D_{2n} is trivial if and only if n is odd. **(2)**

(c) Determine the highest order of an element in the symmetric group S_{10}. **(2)**

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(2) (a) The First Isomorphism Theorem says that there is an isomorphism $G/\ker\Phi \cong \operatorname{im}\Phi$ for any group homomorphism $\Phi : G \longrightarrow H$. Prove this theorem. **(2)**

(b) Use the First Isomorphism Theorem to show that $\mathbb{Z}^2/K \cong \mathbb{Z}_2 \times \mathbb{Z}$, where $K \leq \mathbb{Z}^2$ is the subgroup generated by $(4, 6)$. (*Hint*: Find a surjective group homomorphism $\mathbb{Z}^2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}$ with kernel K.) **(4)**

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(3) When a group acts on itself by conjugation, the orbits are called *conjugacy classes*.

(a) Show that in a finite group, the size of the conjugacy class containing an element a is related to the number of elements commuting with a, i.e., the size of the centralizer, $C_G(a)$. **(2)**

(b) Use the relation to compute the size of the conjugacy class containing the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

in the general linear group $\mathrm{Gl}_2(\mathbb{F}_3)$ of invertible 2×2-matrices over the field with three elements. (Hint: the number of elements in $\mathrm{Gl}_2(\mathbb{F}_3)$ is 48.) **(4)**

PART II - RINGS

(1) (a) Prove that a 2×2-matrix over a field is invertible if and only if the first column is a nonzero vector and the second column is not a multiple of the first column. **(2)**

(b) Let $\mathbb{F}_q$ be a finite field with q elements. Prove that the group $\mathrm{Gl}_2(\mathbb{F}_q)$ of invertible 2×2-matrices over $\mathbb{F}_q$ has $(q^2-1)(q^2-q)$ elements. **(2)**

(c) Determine the number of zero-divisors in the ring $M_2(\mathbb{F}_q)$ of 2×2-matrices over $\mathbb{F}_q$. **(2)**

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(2) (a) Prove that $x^3 - x + 1$ is irreducible in $\mathbb{Z}_3[x]$. **(2)**

(b) Let F be the field $\mathbb{Z}_3[x]/(x^3 - x + 1)$. Write γ for the element $x + (x^3 - x + 1)$, so $F = \mathbb{Z}_3(\gamma)$. Determine the order of γ^2 in the multiplicative group F^*. **(2)**

(c) Let R be the ring $\mathbb{Z}[\sqrt{-3}]$. Is the ideal $(2, 1 + \sqrt{-3})$ a principal ideal in R? **(2)**

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(3) (a) Prove that $f(x) = x^4 + 4x^2 + 2$ is irreducible in $\mathbb{Q}[x]$. **(2)**

(b) Let K be the field $\mathbb{Q}[x]/(f(x))$. Write α for the element $x + (f(x))$, so $K = \mathbb{Q}(\alpha)$. Put $\beta = \alpha^2$. Determine $[\mathbb{Q}(\beta) : \mathbb{Q}]$ and show that $f(x)$ factors as a product of two polynomials of positive degree in $\mathbb{Q}(\beta)[x]$. **(2)**

(c) Prove that $\alpha^3 + 3\alpha$ is a zero of $f(x)$ and conclude that $f(x)$ factors as a product of linear factors in $\mathbb{Q}(\alpha)[x]$. **(2)**