KTH Teknikvetenskap

SF2729 Groups and Rings
Final Exam
Wednesday, August 17, 2011

Time: 14.00-18.00
Allowed aids: none
Examiner: Mats Boij

This final exam consists of two parts; Part I (groups part) and Part II (rings part). The final credit for Part I will be based on the maximum of the results on the midterm exam and Part I in the final exam.

Each problem can give up to 6 points. In the first problem of each part, you are guaranteed a minimum given by the result of the corresponding homework assignment. If you have at least 2 points from HW1, you cannot get anything from Part a) of Problem 1 of Part I, if you have at least 4 points from HW1 you cannot get anything from Part a) or Part b) of Problem 1 of Part I. Similarly for HW2 and Problem 1 of Part II.

The minimum requirements for the various grades are according to the following table:

Grade	**A**	**B**	**C**	**D**	**E**
Total credit	30	27	24	21	18
From Part I	13	12	11	9	8
From Part II	13	12	11	9	8

Present your solutions to the problems in a way such that arguments and calculations are easy to follow. Provide detailed arguments to your answers. An answer without explanation will give no points.

PART I - GROUPS

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(1) (a) A *latin square* of size $n \times n$ is an $n \times n$-array of symbols where each symbol occurs exactly once in each row and in each column. Show that the multiplication table of a finite group has to be a latin square. **(2)**

(b) Let G be the set of invertible 2×2-matrices with coefficients in $\mathbb{Z}_6$. Show that G is a group under matrix multiplication. **(2)**

(c) Lagrange's theorem states that the order of a subgroup H of a finite group G divides the order of G. Prove this theorem. **(2)**

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(2) Let G be the group of invertible 2×2-matrices with entries in $\mathbb{Z}_6$ from problem 1(b) and let G act on $\mathbb{Z}_6 \times \mathbb{Z}_6$ seen as column vectors by matrix multiplication. Let $x = (1, 0) \in \mathbb{Z}_6 \times \mathbb{Z}_6$.

(a) Determine the stabilizer G_x. [1] **(2)**

(b) Determine the orbit Gx. **(2)**

(c) Use the results of part (a) and (b) to determine the order of G. **(2)**

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(3) Let $\Phi\colon G \longrightarrow H$ be a surjective group homomorphism and $K \leq H$ a normal subgroup.

(a) Show that the inverse image $\Phi^{-1}(K)$ is a normal subgroup of G. **(2)**

(b) Show that $G/\Phi^{-1}(K)$ is isomorphic to H/K. **(2)**

(c) Assume that K equals the commutator subgroup $[H, H]$. Show that $\Phi^{-1}(K)$ contains $[G, G]$. Does equality hold? **(2)**

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[1] The stabilizer is also called the *isotropy subgroup*.

PART II - RINGS

(1) (a) Let F be a finite field. Assume that -1 is not a square in F. Prove that 2 or -2 is a square in F. **(2)**

(b) Prove that X^4+1 is irreducible in $\mathbb{Z}[X]$. **(2)**

(c) Let p be a prime number and let $\mathbb{F}_p$ be a finite field with p elements. Prove that X^4+1 is reducible in $\mathbb{F}_p[X]$. (Hint: use part (a) when -1 is not a square in $\mathbb{F}_p$.) **(2)**

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(2) (a) Prove that $3+2i$ is a prime element of $\mathbb{Z}[i]$. **(2)**

(b) Prove that $F=\mathbb{Z}[i]/\mathbb{Z}[i](3+2i)$ is a field. How many elements does F have? **(2)**

(c) Find a generator of the multiplicative group of F. **(2)**

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(3) (a) Prove that the ring $\mathbb{R}[X]/(X^3-X^2+2X-2)$ is isomorphic to $\mathbb{R}\times\mathbb{C}$. **(2)**

(b) Let p be a prime number. Let R be the subring of $\mathbb{Q}$ consisting of the numbers a/b with $a,b\in\mathbb{Z}$ and b not divisible by p. Let I be a nonzero ideal of R. Prove that $I=(p^n)$ for some $n\geq 0$. Conclude that R has a unique maximal ideal. **(4)**