SF2735 Homologisk Algebra
Exercise set 1

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The solutions to these exercises are to be handed in no later than Thursday, 20th of October. Please pay attention to the presentation as well as the arguments given in the solutions.

Exercise 1

(a) (**1 point**). Let $n, m > 1$ be integers. Show that $\hom(\mathbf{Z}/n, \mathbf{Z}/m)$ and $\hom(\mathbf{Z}/m, \mathbf{Z}/n)$ are isomorphic abelian groups.

(b) (**1 point**). Let A and B be finite abelian groups. Show that $\hom(A, B)$ and $\hom(B, A)$ are isomorphic abelian groups.

(c) (**0.5 points**). Give en example of finitely generated abelian groups A and B for which $\hom(A, B)$ and $\hom(B, A)$ are not isomorphic.

Exercise 2 (2 points).

Prove/disprove the following four statements. The $\mathbf{Z}$-module $\mathbf{Q}$ is a) finitely generated, b) free, c) projective, d) injective.

Exercise 3

If A is a finite set, then$|A|$ denotes the number of elements in A. Consider a chain complex C of **finite** abelian groups:

$$0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to 0$$

and the following product:

$$\prod_{i=0}^{n} |C_i|^{-1^i} = |C_0|\,|C_1|^{-1}\,|C_2|\,|C_3|^{-1} \cdots |C_n|^{-1^n}$$

(a) (**2 points or 1 point if you present the solution only for** $n = 2$). Prove that for any i, $H_i(C)$ is a finite abelian group and show the following equality:

$$\prod_{i=0}^{n} |C_i|^{-1^i} = \prod_{i=0}^{n} |H_i(C)|^{-1^i}$$

(b) (**0.5 points**). Show that if C is an exact sequence, then $\prod_{i=0}^{n} |C_i|^{-1^i} = 1$. (you can use (a)).

Exercise 4

(a) (**1 point**). Let A be an abelian group. Show that $\hom(A, \mathbf{Q})$ is a $\mathbf{Q}$ vector space. Prove further that if A is finitely generated, then $\hom(A, \mathbf{Q})$ is a finite dimensional $\mathbf{Q}$ vector space

We can use the above to define:

$$\text{rank}(A) := \dim(\hom(A, \mathbf{Q}))$$

Consider a chain complex C of **finitely generated** abelian groups:

$$0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to 0$$

and the following sum:

$$\sum_{i=0}^{n} -1^i \text{rank}(C_i) = \text{rank}(C_0) - \text{rank}(C_1) + \text{rank}(C_2) - \cdots + (-1)^n \text{rank}(C_0)$$

(b) (**2 points or 1 point if you present the solution only for** $n = 2$). Prove that for any i $H_i(C)$ is a finitely generated abelian group and show the following equality:

$$\sum_{i=0}^{n} -1^i \text{rank}(C_i) = \sum_{i=0}^{n} -1^i \text{rank}(H_i(C))$$

(c) (**0.5 point**). Show that if C is exact, then $\sum_{i=0}^{n} -1^i \text{rank}(C_i) = 0$. (you can use (b)).

(d) (**1 point**). Is it true that if $\sum_{i=0}^{n} -1^i \text{rank}(C_i) = 0$, then C is exact?