SF2735 Homologisk Algebra
Exercise set 2

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The solutions to these exercises are to be handed in no later than Thursday, 3rd of November. Please pay attention to the presentation as well as the arguments given in the solutions.

Exercise 1. (**1 point**) Let A and B be finite abelian groups. Show that the following groups are all isomorphic to each other:

$$\mathrm{hom}(A,B),\ \ \mathrm{hom}(B,A),\ \ A\otimes B,\ \ B\otimes A,\ \ \mathrm{Ext}^1(A,B),\ \ \mathrm{Ext}^1(B,A),\ \ \mathrm{Tor}_1(A,B),\ \ \mathrm{Tor}_1(B,A)$$

Let us denote by the symbol $\mathbf{Ext}(A,B)$ the collection of equivalence classes of short exact sequences $0 \to B \to E \to A \to 0$, where two such sequences are said to be equivalent if they can fit into the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

We use the symbol e to denote the equivalence class of split exact sequences, i.e., the class of sequences which are equivalent to:

$$0 \to B \xrightarrow{\text{in}} B \oplus A \xrightarrow{\text{pr}} A \to 0$$

Exercise 2. Let $\sigma = (0 \to K \to P \to A \to 0)$ be a short exact sequence and $f : K \to B$ be a homomorphism.

(a) (**1 point**) Show that there is a short exact sequence $0 \to B \to E \to A \to 0$ for which the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & A & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

(b) (**1 point**) Show that any two sequences that are solutions to question (a) are equivalent. We denote the equivalence class of such sequences by $f^*(\sigma) \in \mathbf{Ext}(A,B)$.

(c) (**1 point**) Show that if σ and σ' are equivalent exact sequences, then $f^*(\sigma) = f^*(\sigma')$. Conclude that we obtain a map of sets $f^* : \mathbf{Ext}(A,K) \to \mathbf{Ext}(A,B)$.

(d) (**0.5 point**) Show that $f^* : \mathbf{Ext}(A,K) \to \mathbf{Ext}(A,B)$ takes the class of split sequences to the class of split sequences.

Exercise 3. Let $\sigma = (0 \to K \xrightarrow{i} P \to A \to 0)$ be a short exact sequence and $f, g : K \to B$ be homomorphisms.

(a) (**1 point**) Show that $f^*(\sigma) = g^*(\sigma)$ if and only if the homomorphism $f - g : K \to B$ can be expressed as a composition:

$$\overbrace{K \xrightarrow{\;i\;} P \xrightarrow{\;s\;} B}^{f-g}$$

(b) (**1 point**) Let us fix now a short exact sequence $\sigma = (0 \to K \xrightarrow{i} P \to A \to 0)$ for which P is projective. Show that for any element τ in $\mathbf{Ext}(A, B)$, there is a function $f : K \to B$ such that $\tau = f^*(\sigma)$. Conclude that:

$$\hom(K, B) \ni f \mapsto f^*(\sigma) \in \mathbf{Ext}(A, B)$$

is a surjective map of sets.

(c) (**1 point**) Let us fix now a short exact sequence $\sigma = (0 \to K \xrightarrow{i} P \to A \to 0)$ for which P is projective. Show that we have an exact sequence:

$$0 \to \hom(A, B) \to \hom(P, B) \to \hom(K, B) \to \mathrm{Ext}^1(A, B) \to 0$$

Conclude that there is a bijection between $\mathrm{Ext}^1(A, B)$ and $\mathbf{Ext}(A, B)$.

(d) (**0.5 point**) Use the bijection given in the previous statement to define an abelian group structure on $\mathbf{Ext}(A, B)$. Show that e is the zero element in this group structure.

Exercise 4. Let p be a prime number.

(a) (**1 point**) Show that $\mathbf{Ext}(\mathbf{Z}/p^2, \mathbf{Z}/p^2)$ is isomorphic to $\mathbf{Z}/p^2$.

(b) (**1 point**) Let $i : \mathbf{Z}/p^2 \to \mathbf{Z}/p^4$ be the homomorphism that maps the element 1 to the element p^2. Show that $\tau = (0 \to \mathbf{Z}/p^2 \xrightarrow{i} \mathbf{Z}/p^4 \to \mathbf{Z}/p^2 \to 0)$ is a generator of $\mathbf{Ext}(\mathbf{Z}/p^2, \mathbf{Z}/p^2)$. Describe a short exact sequence that is equivalent to $p\tau$.