Homological Algebra and Algebraic Topology

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Introduction

The course is aimed for students who have had some experience with groups, and perhaps seen the definition of rings. We have tried to make these series of lectures self-contained. The theory is often given in terms of commutative rings, but many of the examples are given for the ring of integers. The theoretical result that often simplifies the situation when considering the ring of integers is the Structure Theorem for finitely generated abelian groups. Recall that a finitely generated abelian group M can be written as

$$M = \mathbf{Z}/(n_1) \times \cdots \times \mathbf{Z}/(n_p),$$

for some non-negative integers $0 \leq n_1 \leq n_2 \cdots \leq n_p$. This provides a description of finitely generated modules over the integers, a description we use to give examples of the general theory and of general definitions.

The purpose of the second part of these notes concerning topological spaces is to introduce the basic concepts of topological spaces and their homology groups. We stress examples and tools to compute these invariants. In particular we compute the homology of spheres and projective spaces.

We do not claim any originality with the text. We have freely used the existing literature as reference. In particular, for the first six chapters we did use Rotman, "Introduction to Homological Algebra" as guidance. That book was in particular useful for the technical proofs dealing with resolutions and the independence of the choices involved.

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Chapter 1

Modules

1.1 Rings

Groups considered here are abelian and denoted as $(A,+)$ where 0 denotes the identity element. A ring A will in these notes always mean a commutative unital ring. Recall that a ring A is an abelian group $(A,+)$ with an additional product structure which is compatible with the group structure. The product is denoted with $\cdot$, but is in many cases simply suppressed. In any case it is a map of sets

$$A \times A \longrightarrow A$$

which is associative, $a\cdot(b\cdot c) = (a\cdot b)\cdot c$, and distributive, $a\cdot(b+c) = a\cdot b+a\cdot c$, for all elements a, b and c in A. The ring being commutative means that $a \cdot b = b \cdot a$, for all a, b in A. Unital means that there exists an element 1 in A such that $1 \cdot a = a$ for all a.

The typical examples of rings will for us be the integers $A = \mathbf{Z}$ with the usual addition and multiplication, and fields.

1.1.1 Definition. *A commutative, non-zero, and unital ring A is a* field *if any non-zero element $a \in A$ has a multiplicative inverse a^{-1} in A, that is an element such that $a \cdot a^{-1} = 1$.*

Examples of fields are the rational numbers $\mathbf{Q}$, the real numbers $\mathbf{R}$ and the complex numbers $\mathbf{C}$. Other fields we have in mind are finite fields $\mathbf{F}_p$ obtained as quotients of $\mathbf{Z}$ by the subgroup generated by some prime number p.

Even though our statements usually are given in terms of general commutative rings, we have our focus on the ring of integers $\mathbf{Z}$ and fields, only.

1.2 Modules

1.2.1 Definition. *Let A be a ring and $(M,+)$ an abelian group. Assume furthermore that we have a map of sets $A \times M \longrightarrow M, (a,x) \mapsto ax$, such that*

$$a(x+y) = ax + ay \qquad (a+b)x = ax + bx$$
$$(ab)x = a(bx) \qquad 1x = x$$

for all elements a, b in A, and all elements x, y in M. Such a map gives the abelian group $(M,+)$ an A-module structure. We will refer to such a map by saying that M is an A-module.

1.2.2 Example. Any abelian group $(M,+)$ is in a natural way a $\mathbf{Z}$-module. We have the natural map

$$\mathbf{Z} \times M \longrightarrow M$$

sending (n,x) to $n \cdot x = x + \cdots + x$. This defines a $\mathbf{Z}$-module structure. Consequently, each abelian group has a natural $\mathbf{Z}$-module structure.

1.2.3 Example. Let A be a field. In that case a module M over A is a vector space, and the map defining the A-module structure on the abelian group is the usual scalar multiplication. Thus, the notion of a module is a generalization of vector spaces.

1.2.4 Example. Let $(A,+)$ be the abelian group structure of the ring A. Together with the product structure on A we have that A is an A-module. In fact, any ideal $I \subseteq A$ is an A-module. Recall that a subgroup $(I,+) \subseteq (A,+)$ such that $ax \in I$, for any $a \in A$ and any $x \in I$, is called an *ideal*. Thus, for any ideal $I \subseteq A$ we have that the product structure from the ring gives a map

$$A \times I \longrightarrow I,$$

satisfying the criteria of (1.2.1). Hence, we have that any ideal is in a natural way an A-module. From an ideal I in a commutative ring we can always construct the quotient ring A/I. The quotient ring is also in a natural way an A-module.

1.2.5 Example. Let $\varphi : A \longrightarrow B$ be a homomorphism of rings, that is a map of the underlying sets such that $\varphi(a+b) = \varphi(a) + \varphi(b), \varphi(ab) = \varphi(a)\varphi(b)$ and where $\varphi(1) = 1$. Then the ring B can be viewed as an A-module in the obvious way. More generally, let M be a B-module. Then we have a natural map

$$A \times M \longrightarrow M$$

taking (a,x) to $\varphi(a)x$. It is straightforward to check that the map gives M the structure of an A-module. We say that the B-module M becomes an A-module by restriction of scalars.

1.3 Submodules and quotients

1.3.1 Definition. *Let M be an A-module, and let $(N, +) \subseteq (M, +)$ be a subgroup. If the A-module structure on M restricts to an A-module structure on N, that is, we get by restriction an induced map*

$$A \times N \longrightarrow N,$$

then we say that N is a submodule of M.

Note that a non-empty subset $N \subseteq M$ which is closed under addition, and under multiplication by elements from A, will be a submodule.

1.3.2 Lemma. *Let N be a submodule of M. Then the quotient group $(M/N, +)$ has an induced A-module structure*

$$A \times M/N \longrightarrow M/N$$

sending $(a, \overline{x})$ to $\overline{ax}$, where $\overline{z}$ denotes the equivalence class of z in M/N, for $z \in M$.

Proof. Let us first show that the description above indeed gives a well-defined map. Let x and y be two elements in M that represent the same equivalence class in M/N. We need to see that the equivalence classes of ax and ay are equal, for any $a \in A$. Since x and y represent the same equivalence class in M/N we have that $x = y + n$ in M for some $n \in N$. Then $ax = ay + an$ in M, and an is in N. It follows that $\overline{ax} = \overline{ay}$ in M/N. Thus there is a well-defined map, and it is readily checked that it defines an A-module structure on M/N. □

1.4 Basis

Let $x_1, \ldots, x_m$ be m elements in an A-module M. Consider the subset

$$< x_1, \ldots, x_m >= \{a_1x_1 + \ldots + a_mx_m \mid a_i \in A\} \subseteq M.$$

1.4.1 Lemma. *The set $< x_1, \ldots, x_m >$ is an A-submodule of M.*

Proof. The set is closed under addition and also under multiplication by elements from A. Namely, if $x \in< x_1, \ldots, x_m >$, then we have that there are scalars $a_1, \ldots, a_m$ in A such that

$$x = a_1x_1 + \cdots + a_mx_m.$$

For any scalar $a \in A$ we have (1.2.1)

$$ax = aa_1x + \cdots + aa_mx_m,$$

hence the set is a submodule of M. $\square$

We refer to this as the submodule *generated* by the elements $x_1, \ldots, x_m$. If the submodule generated by $x_1, \ldots, x_m$ equals the ambient module M, then we say that $x_1, \ldots, x_m$ *span* the A-module M.

1.4.2 Definition. *Let M be an A-module. If there exists a finite set of elements $x_1, \ldots, x_m$ in M that spans the module, then we say that the module M is finitely generated.*

1.4.3 Example. The integer plane $M = \mathbf{Z} \times \mathbf{Z}$ is a finitely generated $\mathbf{Z}$-module. A set of generators is $x_1 = (1, 0), x_2 = (0, 1)$. Indeed, any element (m, n) in M can be written as the sum

$$(m, n) = mx_1 + nx_2.$$

Hence the two elements x_1 and x_2 span M.

1.4.4 Example. The Fundamental Theorem for finitely generated Abelian groups says that an abelian group is finitely generated if and only if it is of the form

$$M = \mathbf{Z} \times \cdots \times \mathbf{Z} \times \mathbf{Z}/(n_1) \times \cdots \times \mathbf{Z}/(n_r),$$

where $n_i \geq 1$. Thus a finitely generated abelian group contains a finite number of copies of $\mathbf{Z}$, and a torsion part which is the product of a finite number of cyclic groups of finite order. If we let $x_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \in M$ denote the element having the class of 1 at the i'th component, then these elements span the module M.

Linear Independence

We say that the elements $x_1, \ldots, x_m$ in a module M are linearly independent, over A, if

$$\sum_{i=1}^{m} a_ix_i = 0 \Rightarrow \{a_i = 0, \quad \text{for all} \quad i = 1, \ldots, m.\}$$

Finally, we say that the *ordered* sequence $x_1, \ldots, x_m$ form an A-module *basis* of M if the elements span M and are linearly independent. Note that if $x_1, \ldots, x_m$ is an ordered sequence that form an A-module basis of M, then any reordering of the basis will still be linearly independent and still span M. However, generally a reordering of a basis will give a different basis.

1.4.5 Definition. *A module M that admits a basis is called a* free *module.*

1.4.6 Remark. The zero module $M = 0$ is a free module. The basis is the empty set.

1.4.7 Example. The elements x_1 and x_2 of Example (1.4.3) will form a basis.

1.4.8 Example. One major difference between modules and vector spaces is that modules do not always have a basis. For instance, the field $\mathbf{F}_2$ of two elements is a quotient of $\mathbf{Z}$, and in particular a $\mathbf{Z}$-module. The element $1 \in \mathbf{F}_2$ spans the module, whereas the zero element does not. The element 1 is however not a basis, as for instance $0 \cdot 1 = 2 \cdot 1$ in $\mathbf{F}_2$.

1.4.9 Example. The elements $\{x_1, \dots, x_n\}$ of the Example (1.4.4) will in in general not form a basis. In fact it will be a basis if and only if the group has no torsion part.

1.4.10 Proposition. *Let M be an A-module that admits a basis $x_1, \dots, x_m$. Then the any other basis of M will have the same number of elements.*

Proof. Exercise (1.8.11). □

1.4.11 Definition. *An A-module M which has a basis consisting of m elements is said to be* free of rank *m.*

1.4.12 Example. For vector spaces we have that any non-zero element can be extended to be part of some basis. This is not true for modules, not even when the module does have a basis. Consider the ring of integers $\mathbf{Z}$ as a module over itself. The element 1 will form a basis, as well as the element -1. However, any other non-zero element x will be linearly independent, but the element will not span $\mathbf{Z}$. For instance, in the free rank one module $\mathbf{Z}$ we have the descending chain of proper submodules $\mathbf{Z} \supset (2) \supset (4) \supset (8) \cdots$

1.5 Direct sum

Let $\{M_i\}_{i \in I}$ be a collection of A-modules, indexed by some set I. The product $\times_{i \in I} M_i$ consisting of all sequences of elements $(x_i)_{i \in I}$ with $x_i \in M_i$ is an abelian group by componentwise addition. The product becomes naturally also an A-module via

$$(a, (x_i)_{i \in I}) \mapsto (ax_i)_{i \in I},$$

for all $a \in A$, and all sequences $(x_i)_{i \in I}$ in the product.

1.5.1 Definition. *Let A be a ring, and let $\{M_i\}_{i\in I}$ be a collection of A-modules. The direct sum $\oplus_{i\in I} M_i$ is defined as the A-submodule*

$$\oplus_{i\in I} M_i \subseteq \times_{i\in I} M_i$$

consisting of sequences (x_i) where all but a finite number of elements are zero.

1.5.2 Remark. Note that in general the direct sum is a proper submodule of the product. However, when the indexing set I is finite, then the direct sum equals the product.

1.5.3 Example. It is interesting to consider the direct sum of copies of A, that is $M_i = A$ for $i = 1, \ldots, n$. The module $M = \oplus_{i=1}^n A$ contains the n elements

$$e_i := (0, 0, \ldots, 0, 1, 0, \ldots, 0),$$

where 1 appears on the i'th component for $i = 1, \ldots, n$. From the definition of the module $M = \oplus_{i=1}^n A$ we have that any element $x \in M$ is of the form $x = (x_1, \ldots, x_n)$, with $x_i \in A$. We can therefore write

$$x = (x_1, \ldots, x_n) = x_1 e_1 + \cdots + x_n e_n.$$

Thus any element $x \in M$ is an A-linear combination of the elements $e_1, \ldots, e_n$, and moreover the linear combination is unique. The elements $e_1, \ldots, e_n$ form an A-module basis of M. In particular the ring A viewed as a module over itself has the basis 1.

1.6 Homomorphisms

Having defined the notion of an A-module, we will need to define the notion of maps between these objects.

1.6.1 Definition. *Let M and N be two A-modules. An A-module homomorphism is a map of the underlying sets $f\colon M \longrightarrow N$, such that*

$$f(ax + by) = af(x) + bf(y)$$

for all a, b in A, and all x, y in M.

Note that an A-module homomorphism $f\colon M \longrightarrow N$ is exactly the same as a group homomorphism that is compatible with the A-module structure.

Kernel and Image

The *kernel* of an A-module homomorphism $f\colon M \longrightarrow N$ is the set

$$\operatorname{Ker}(f) = \{x \in M \mid f(x) = 0\}.$$

Any module has the zero element $0 \in M$, and the zero element is in the kernel of any A-module homomorphism. An A-module homomorphism f is said to be *injective* if the kernel is trivial, that is $\operatorname{Ker}(f) = 0$. The image is the set

$$\operatorname{Im}(f) = \{f(x) \mid x \in M\} \subseteq N.$$

The homomorphism is surjective if the image equals all of N. An A-module homomorphism $f\colon M \to N$ is an isomorphism if both injective and surjective.

1.6.2 Lemma. *An A-module M is free if and only if it is isomorphic to $\oplus_{i \in I} A$, for some (possibly empty) indexing set I.*

Proof. See Exercise (1.8.19). □

1.6.3 Lemma. *Let $f\colon M \longrightarrow N$ be an A-module homomorphism. Then we have the following.*

(1) The kernel $\operatorname{Ker}(f)$ is an A-submodule of M.

(2) The image $\operatorname{Im}(f)$ is an A-submodule of N.

(3) The homomorphism f has a unique factorization through an injective A-module homomorphism $\overline{f}\colon M/\operatorname{Ker}(f) \longrightarrow N$.

In particular we have that if $f\colon M \longrightarrow N$ is a surjective A-module homomorphism, then $\overline{f}\colon M/\operatorname{Ker}(f) \longrightarrow N$ is an isomorphism.

Proof. Readily checked. □

For any A-module M we have the identity morphism $\mathrm{id}_M\colon M \longrightarrow M$ sending x to x, for all $x \in M$.

1.6.4 Lemma. *Let $f\colon M \longrightarrow N$ be an A-module isomorphism. Then there exists a unique A-module homomorphism $g\colon N \longrightarrow M$ such that $gf = id_M$ and $fg = id_N$.*

Proof. There is a set-theoretic map $g\colon N \longrightarrow M$ having the property that $g(f(x)) = x$ and $f(g(y)) = y$ for all $x \in M$, all $y \in N$. One has to verify that this map is in fact an A-module homomorphism. □

1.6.5 Proposition. *A module M is finitely generated if and only if the module can be written as a quotient module of a free module of finite rank.*

Proof. Note that the image of an A-module homomorphism $f\colon \oplus_{i=1}^m A \longrightarrow M$ is exactly the same as the submodule generated by $f(e_1), \dots, f(e_m)$. Thus, m elements in M that span the module M is exactly the same as having a surjective map $f\colon \oplus_{i=1}^m A \longrightarrow M$. □

The set of all A-module homomorphisms from an A-module M to another A-module N is denoted

$$\mathrm{Hom}_A(M, N).$$

If $M = N$ then a homomorphism is referred to as an endomorphism, and the set of all endomorphisms is denoted as $\mathrm{End}_A(M)$.

1.6.6 Lemma. *Let M and N be two A-modules. The set* $\mathrm{Hom}_A(M, N)$ *becomes an abelian group with pointwise addition, and has the structure of an A-module by the map*

$$A \times \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_A(M, N)$$

sending (a, f) to the map $M \longrightarrow N$ that takes x to $af(x)$.

Proof. Readily checked. □

1.6.7 Proposition. *Let $M_1, \dots, M_n$ be A-modules, and let $M = \oplus_{i=1}^n M_i$ denote their direct sum. For any A-module N we have a natural isomorphism*

$$\mathrm{Hom}_A(\oplus_{i=1}^n M_i, N) = \oplus_{i=1}^n \mathrm{Hom}_A(M_i, N).$$

Thus, giving an A-module homomorphism from a finite direct sum is equivalent to giving A-module homomorphisms from the components appearing in the direct sum.

Proof. There is an A-module map $c_k\colon M_k \longrightarrow M = \oplus_{i=1}^n M_i$ sending elements $x \mapsto (0, 0, \dots, 0, x, 0, \dots, 0)$ to the k'th component, for $k = 1, \dots, n$. Any morphism $f\colon M \longrightarrow N$ composed with c_k give an A-module homomorphism $f \circ c_k\colon M_k \longrightarrow N$. We then get an A-module homomorphism

$$\Phi\colon \mathrm{Hom}_A(M, N) \longrightarrow \oplus_{i=1}^n \mathrm{Hom}_A(M_i, N) \tag{1.1}$$

taking $f \mapsto (f \circ c_1, \dots, f \circ c_n)$. Let us now construct the morphism going the other way. Let $(f_1, \dots, f_n)$ be an element of $\oplus_{i=1}^n \mathrm{Hom}_A(M_i, N)$. Then each

f_k is an A-module homomorphism $M_k \longrightarrow N$, and we define the A-module homomorphism $f\colon M \longrightarrow N$ by

$$f(x_1, \ldots, x_n) := f_1(x_1) + \cdots + f_n(x_n).$$

One checks that since each f_k is an A-module homomorphism, then also the map f is an A-module homomorphism. We have then constructed an A-module homomorphism Ψ which we need to check is the inverse of (1.1). Let $f\colon M \longrightarrow N$ be an A-module morphism, and consider the A-module homomorphism $\Psi(\Phi(f))\colon M \longrightarrow N$. We want to show that these two homomorphisms are equal, which means that they have exactly the same value on every element $x \in M$. We have that

$$\Psi(\Phi(f))(x) = \Psi(f \circ c_1(x), \ldots, f \circ c_n(x)) = f \circ c_1(x) + \cdots + f \circ c_n(x).$$

As $x \in M$ we have that $x = (x_1, \ldots, x_n)$, with $x_k \in M_k$, for all $k = 1, \ldots, n$. Consequently, we have that

$$\begin{aligned} f \circ c_1(x) + \cdots + f \circ c_n(x) &= f(x_1, 0, \ldots, 0) + \cdots + f(0, \ldots, 0, x_n) \\ &= f(x_1, \ldots, x_n) = f(x). \end{aligned}$$

Hence $\Psi(\Phi)$ is the identity. One checks in a similar way that $\Phi(\Psi)$ also equals the identity, and thereby completes the proof of the proposition. $\square$

1.6.8 Corollary. *Let M be an A-module with basis $e_1, \ldots, e_m$ and N an A-module with basis $f_1, \ldots, f_n$. Then the A-module $\mathrm{Hom}_A(M, N)$ is isomorphic to the A-module of $(n \times m)$-matrices with coefficients in A.*

Proof. As M has basis $e_1, \ldots, e_m$ we can by Lemma (1.6.2) identify M with $\oplus_{i=1}^m A$. Consequently, by the proposition, we have

$$\mathrm{Hom}_A(M, N) = \oplus_{i=1}^m \mathrm{Hom}_A(A, N).$$

Any A-module homomorphism $f\colon A \longrightarrow N$ is determined by its value on the element $1 \in A$, and moreover any element in N would give rise to an A-module homomorphism. It follows that $\mathrm{Hom}_A(A, N) = N$. As N has basis $f_1, \ldots, f_n$ we have that $N = \oplus_{i=1}^n A$. Consequently $\mathrm{Hom}_A(A, N)$ equals the A-module of ordered n-tuples of elements in A, and we have that $\mathrm{Hom}_A(M, N)$ equals the A-module of ordered $m \cdot n$-tuples of elements in A. This is the same as $(n \times m)$-matrices with coefficients in A. $\square$

Matrix notation

It is customary to represent an A-module homomorphism $f\colon \oplus_{i=1}^m A \longrightarrow \oplus_{i=1}^n A$ with an $(n \times m)$ matrix X_f where the k'th column is the coefficient of $f(0, \ldots, 0, 1, 0, \ldots, 0)$. An element $x \in \oplus_{i=1}^m A$ is then viewed as a column vector $[x] := [x_1, \ldots, x_m]^{tr}$, and the action of f on this particular element is then given by the matrix multiplication $X_f \cdot [x]$.

1.7 Determinants

Consider now the situation with an A-module M that has a basis $e_1, \ldots, e_m$. Let $f\colon M \longrightarrow M$ be an endomorphism, which we can by (1.6.8) represent uniquely with an $(m \times m)$-matrix

$$X_f = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,m} \end{bmatrix},$$

with coefficients $a_{i,j} \in A$. We define the determinant of the endomorphism $f\colon M \longrightarrow M$ to be the determinant of the corresponding matrix representation,

$$\det(f) := \sum_{\sigma \in \mathfrak{S}_m} (-1)^{|\sigma|} a_{1,\sigma(1)} \cdots a_{m,\sigma(m)}.$$

Note that $\det(f)$ is an element of the ring A.

1.7.1 Proposition. *Let $f\colon M \longrightarrow M$ be an A-module endomorphism of an A-module that has a finite basis. Then f is an isomorphism if and only if* $\det(f)$ *is invertible in A.*

Proof. If f were an isomorphism it would be invertible. On the level of matrices there would exist a matrix Y such that the product $X_f Y$ would equal the identity matrix I_m. The determinant of the identity matrix is 1, and we would get

$$1 = \det(X_f Y) = \det(f) \cdot \det(Y) \quad \in A.$$

In other words $\det(f)$ is invertible in A. To prove the converse, let $\operatorname{ad}(M)$ denote the adjoint matrix of X_f. The adjoint matrix is the transpose of the cofactor matrix, where the coefficient (i, j) of the cofactor matrix is $(-1)^{i+j} c_{i,j}$, with $c_{i,j}$ being the determinant of the matrix X_f with row i and column j

removed. Note that the adjoint matrix has coefficients in A, and Cramer's identity yields the following matrix equality

$$X_f \cdot \mathrm{ad}(X) = \det(f) \cdot I_m.$$

As $\det(f)$ was assumed invertible we get that

$$\det(f)^{-1}\mathrm{ad}(X)$$

is the matrix inverse of X_f. Hence $f\colon M \longrightarrow M$ has an inverse, and is consequently an isomorphism. $\square$

1.8 Exercises

1.8.1. Show that the fields $\mathbf{R}$ and $\mathbf{C}$ are not isomorphic.

1.8.2. Show that the multiplicative inverse of an element x in a commutative unital ring A is unique, if it exists.

1.8.3. Solve the equation $3x = 2$ in $\mathbf{Z}/(7)$.

1.8.4. Describe all ideals I in the ring $A = \mathbf{Z}/(12)$, and in each case describe the quotient ring A/I.

1.8.5. The *sum* of two subvector spaces V_1 and V_2 of a vector space V is defined as

$$V_1 + V_2 = \{x_1 + x_2 \mid x_i \in V\} \subseteq V.$$

The sum is again a vector space. Show that

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2).$$

1.8.6. Consider an inclusion of A-modules $L \subseteq N \subseteq M$. Show that we have the inclusion $N/L \subseteq M/L$, and that we have

$$(M/L)/(N/L) = M/N.$$

1.8.7. Let A be a commutative unital ring. Assume that A contains no zero-divisors other than the zero element $0 \in A$, and assume that the underlying set $|A|$ is finite. Show that A is in fact a field. (An element $x \in A$ is a *zero-divisor* if there exists a non-zero element y such that $xy = 0$).

1.8.8. Let $\varphi\colon A \longrightarrow B$ be a homomorphism of rings.

(1) Show that if $J \subseteq B$ is an ideal, then $\varphi^{-1}(J)$ is an ideal in A.

(2) Let $I \subseteq A$ be an ideal. Show that if φ is surjective, then $\varphi(I)$ is an ideal in B.

(3) Show, by an example, that $\varphi(I)$ is not necessary an ideal, even if I is an ideal.

1.8.9. Verify that $a0 = 0$ and that $(a - b)x = ax - bx$ for any elements a, b in a ring A, and any x in an A-module M.

1.8.10. Let V be a vector space over some field F, and let $F[X]$ denote the polynomial ring in one variable X over F. Show that a $F[X]$-module structure on V is equivalent with having a F-linear map $T\colon V \longrightarrow V$.

1.8.11. Let E be a free A-module that admits a basis $\beta = e_1, \dots, e_n$. Show that any other basis of E will have the same cardinality as β.

1.8.12. Let E be a free A-module of finite rank. Show that $\mathrm{End}_A(E)$ is free A-module, and find a basis.

1.8.13. Let M be an A-module, and let $M^* = \mathrm{Hom}_A(M, A)$ denote its *dual*. Show that there is a natural A-module homomorphism $M \longrightarrow (M^*)^*$ from a module to its double dual. Show, furthermore, that if M is free of finite rank, then M is isomorphic to its double dual.

1.8.14. Let M be an A-module. Show that the A-module $\mathrm{End}_A(M)$ has a ring (not necessarily commutative) structure by composition.

1.8.15. A non-zero A-module M is called simple if the only submodules are the zero module and M itself. Thus a module M is simple if the trivial submodules are the only submodules of M.

(1) Show that a simple module is isomorphic to $A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ in A.

(2) Show Schur's Lemma: Let $f\colon M \longrightarrow N$ be an A-module homomorphism between two simple modules. Show that f is either the zero map or an isomorphism.

(3) Show that if M is simple, then $\mathrm{End}_A(M)$ is a skew-field; that is, a not necessarily commutative field.

1.8.16. The sum of two submodules M and N of an A-module P is defined similarily as with vector spaces (1.8.5). Show that

$$(M + N)/N = N/(M \cap N).$$

1.8.17. Let M be an A-module, and $I \subseteq A$ an ideal generated by $x_1, \dots, x_n$. Assume that for each $i = 1, \dots, n$ there exists an integer p_i such that $x_i^{p_i}$ annihilates M. Show that there exists an integer m such that I^m annihilates M. (An element $a \in A$ *annihilates* an element $m \in M$ if $am = 0$, and the element $a \in A$ annihilates M if $am = 0$ for all $m \in M$).

1.8.18. Let $I \subseteq A$ be a non-zero ideal. Show that I is a free A-module if and only if I is principally generated by an element which is not a zero-divisor. (An ideal I is *principally* generated if it is generated by one element).

1.8.19. Prove Lemma (1.6.2).

1.8.20. Show that the $\mathbf{Z}$-submodule $\mathbf{Z}[p] \subseteq \mathbf{Q}$ consisting of all integer coefficient polynomial expressions in p, is not finitely generated unless $p \in \mathbf{Z}$.

1.8.21. An element x in an A-module M is a *torsion* element if there exists an element $a \in A$ that is not a zero-divisor and annihilates x (1.8.17). Show that the torsion elements in M form an A-submodule, the torsion module of M.

1.8.22. Show that $\mathbf{Q}$ is torsion free, but not free as an $\mathbf{Z}$-module.

1.8.23. Show that $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/(m), \mathbf{Z}) = 0$ for any non-zero integer $m \neq 0$.

1.8.24. Show that $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/(m), \mathbf{Z}/(n)) = \mathbf{Z}/(d)$, where d is the greatest common divisor of m and n.

1.8.25. Let

$$M = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \end{bmatrix},$$

where the coefficients in the matrix are in $\mathbf{Z}/(30)$. Show that the rows of M are linearly independent over $\mathbf{Z}/(30)$, but any two columns are linearly dependent (a situation which does not occur for vector spaces).

1.8.26. The integer plane $\mathbf{Z}^2$ consists of all ordered pairs of integers $v = (n, m)$. We have seen (Example (1.4.7)) that the integer plane is a free $\mathbf{Z}$-module of rank two.

1. Let $v = (m, n)$ be a vector in the integer plane $\mathbf{Z}^2$. Show that v is part of a basis if and only if $(m, n) = 1$ (the integers are coprime).
2. Let $v = (5, 12)$. Extend u to a basis of the integer plane.

1.8.27. Show that $\mathbf{Z}/(m) \oplus \mathbf{Z}/(n) = \mathbf{Z}/(nm)$ if and only if $(n, m) = 1$.

1.8.28. Find an example where $M \oplus P = N \oplus P$, but where M is not isomorphic to N.

1.8.29. A ring A is noetherian if any ascending chain of ideals $I_1 \subseteq I_2 \subseteq \cdots$ becomes stationary. Show that the ascending chain condition is equivalent with any submodule of a finitely generated module being finitely generated. In particular any ideal in a noetherian ring is finitely generated. As the ideals in $\mathbf{Z}$ are principally generated, we have that $\mathbf{Z}$ is a noetherian ring.

Chapter 2

Complexes

2.1 Complexes

Let $P_\bullet = \{P_n, d_n\}_{n\in\mathbf{Z}}$ be a family of modules P_n, and module homomorphisms

$$d_n\colon P_n \longrightarrow P_{n-1}$$

indexed by the integers. The module P_n is the *degree n* component of the family, and the homomorphism d_n is the *boundary operator* in degree n. Note that the boundary operators are decreasing the degree. We will eventually also allow these boundary operators to be increasing, but as default we consider decreasing boundary operators.

A collection $\{P_n, d_n\}$ with decreasing boundary operators is called a *complex* if $d_{n-1} \circ d_n = 0$, for all n. And a collection $\{P_n, d_n\}$ with increasing boundary operators is called a complex if $d_{n+1} \circ d_n = 0$, for all n.

The issue with ascending versus descending boundary operators is a matter of indexing, the important property is that the composition of any two adjacent boundary operators equals the zero map.

2.1.1 Lemma. *The composition of two module homomorphisms $f\colon M \longrightarrow N$ and $g\colon N \longrightarrow P$ is the zero homomorphism if and only if*

$$\mathrm{Im}(f) \subseteq \mathrm{Ker}(g).$$

In particular, a family $\{P_n, d_n\}$ of modules a homomorphisms, is a complex if and only if

$$\mathrm{Im}(d_{n+1}) \subseteq \mathrm{Ker}(d_n),$$

for all n.

Proof. This is just a restatement. □

2.1.2 Example. Any module M can be viewed as a complex

$$\cdots \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots .$$

It is customary to put the module M in degree zero, unless otherwise mentioned. So M is the complex $\{P_n, d_n\}$ where $P_0 = M$, and all the other are zero modules. The boundary operators d_n are all the zero map.

2.1.3 Example. The sequence of $\mathbf{Z}$-modules

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{Z} \xrightarrow{4} \mathbf{Z} \longrightarrow \mathbf{Z}/(2) \longrightarrow 0 \longrightarrow \cdots$$

is a complex. The first non-trivial map is multiplication by 4, and the second non-trivial map is the quotient map. The multiplication by a non-zero integer $n\colon \mathbf{Z} \longrightarrow \mathbf{Z}$ is always injective, and the image of such a map is the submodule $(n) \subseteq \mathbf{Z}$ generated by n. The kernel of the projection morphism $\mathbf{Z} \longrightarrow \mathbf{Z}/2$ is the submodule (2). We then have that in the sequence above we could have replaced 4 with any integer of the form $2n$, and the sequence would still be a complex.

2.1.4 Remark. Strictly speaking a complex is indexed by the integers, but often we encounter a finite collection of modules and homomorphism. We will in such cases tacitly assume that the other modules are zero modules. In particular if we have a sequence of A-modules and A-module homomorphisms

$$P_n \xrightarrow{d_n} \cdots \xrightarrow{d_k} P_{k-1} ,$$

where $\mathrm{Im}(d_p) \subseteq \mathrm{Ker}(d_{p-1})$ for all $n \leq p < k$, we will refer to the sequence being a complex. This means that we assume that $P_m = 0$ for $m \geq n$ and for $m \leq k-2$.

2.1.5 Definition. *A complex* $\{P_n, d_n\}$ *is an* exact sequence *if* $\mathrm{Im}(d_{n+1}) = \mathrm{Ker}(d_n)$ *for all* n*. Furthermore, an exact sequence with at most three non-trivial elements is a* short exact sequence*, and written as*

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0.$$

2.1.6 Example. The complex (2.1.3) is not exact, but

$$0 \longrightarrow \mathbf{Z} \xrightarrow{4} \mathbf{Z} \longrightarrow \mathbf{Z}/(4) \longrightarrow 0$$

is short exact.

2.1.7 Example. Let $f\colon M \longrightarrow N$ be an A-module homomorphism. The cokernel of a homomorphism f is the quotient module $N/\mathrm{Im}(f)$, and usually denoted $\mathrm{Coker}(f)$. We can then form the sequence

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow M \xrightarrow{f} N \longrightarrow \mathrm{Coker}(f) \longrightarrow 0$$

which we consider being an complex with other terms being zero modules and suppressed, and where the degrees also are suppressed. The complex is exact by construction.

2.1.8 Example. Note that a sequence $M \xrightarrow{f} N \longrightarrow 0$ is exact is equivalent with f being surjective. A sequence $0 \longrightarrow M \xrightarrow{f} N$ is exact is equivalent with f being injective. Exactness of the sequence

$$0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$$

is another way of saying that f is an isomorphism.

2.2 Chain maps

Consider the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "d"'] & M' \arrow[d, "d'"] \\ N \arrow[r, "g"] & N' \end{tikzcd}$$

of A-modules and A-module homomorphisms. Such a diagram is referred to as being *commutative* if $d' \circ f = g \circ d$.

2.2.1 Definition. *A chain map $f\colon P_\bullet \longrightarrow Q_\bullet$ is a collection of A-module homomorphisms $f_n\colon P_n \longrightarrow Q_n$ commuting with the boundary operators. In other words we have commutative diagrams*

$$\begin{tikzcd} P_n \arrow[r, "f_n"] \arrow[d, "d_n^P"] & Q_n \arrow[d, "d_n^Q"] \\ P_{n-1} \arrow[r, "f_{n-1}"] & Q_{n-1} \end{tikzcd}$$

for every n.

2.2.2 Example. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbf{Z} \arrow[r, "4"] \arrow[d] & \mathbf{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & \mathbf{Z}/(2) \arrow[r] & 0.
\end{tikzcd}$$

Each horizontal row we think of representing a complex. As the diagrams commute we have that the diagram represents a chain map of complexes.

2.2.3 Proposition (Snake Lemma)**.** *Let*

$$\begin{tikzcd}
& M_1 \arrow[r, "\varphi_1"] \arrow[d, "f_1"] & M_2 \arrow[r, "\varphi_2"] \arrow[d, "f_2"] & M_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & N_1 \arrow[r, "\psi_1"] & N_2 \arrow[r, "\psi_2"] & N_3 &
\end{tikzcd}$$

be a commutative diagram of A-modules and A-module homomorphisms, where the two horizontal sequences are exact. Then we have a long exact sequence of A-modules

$$\begin{tikzcd}
\mathrm{Ker}(f_1) \arrow[r] & \mathrm{Ker}(f_2) \arrow[r] & \mathrm{Ker}(f_3) \arrow[dll, "\delta"] \\
\mathrm{Coker}(f_1) \arrow[r] & \mathrm{Coker}(f_2) \arrow[r] & \mathrm{Coker}(f_3)
\end{tikzcd} \quad . \tag{2.1}$$

The two first and the two last homomorphisms are the obvious ones, whereas the connecting homomorphism δ takes an element $x \in \mathrm{Ker}(f_3)$ to the class of

$$\psi_1^{-1}(f_2(\varphi_2^{-1}(x))) \in \mathrm{Coker}(f_1).$$

Furthermore, if φ_1 is injective, then the left most homomorphism in the long exact sequence is injective. If ψ_2 is surjective, then the rightmost homomorphism in the long exact sequence is surjective.

Proof. It is readily checked that we have an induced homomorphism of kernels and cokernels, that is the two first and the two last arrows of the sequence displayed above. We will describe the particular map $\delta\colon \mathrm{Ker}(f_3) \longrightarrow \mathrm{coker}(f_1)$. Let $x \in M_3$ be an element mapped to zero by f_3, and let $x' \in M_2$ be any element such that $\varphi_2(x') = x$. By commutativity of the diagram we have $\psi_2(f_2(x')) = f_3(x)$, hence $f_2(x') \in \mathrm{Ker}(\psi_2)$. Consequently there is a unique element $x'' \in N_1$ such that $\psi_1(x'') = f_2(x')$. Let $\delta(x)$ be the image of x'' by the quotient map $N_1 \longrightarrow \mathrm{coker}(f_1)$. This prescribed map involved a choice of pre-image x' of x, and we need to check that we indeed get a well-defined map. Let $y' \in M_2$ be another pre-image of x. Then $\varphi_2(x' - y') = 0$. By exactness of the upper row there exist $z \in M_1$ such that $\varphi_1(z) = x' - y'$. By the commutativity of the left diagram it follows that the two different pre-images x' and y' give the same element in $\mathrm{coker}(f_1)$. Consequently we have a well-defined map $\delta \colon \mathrm{Ker}(f_3) \longrightarrow \mathrm{coker}(f_1)$, which is easily seen to be an A-module homomorphism.

We furthermore leave to the reader to check that we not only have obtained a complex in (2.1), but that it is actually an exact sequence. The only part we will go through in detail is exactness at one particular step. We will show that $\mathrm{Ker}(\delta)$ equals the image of the induced map $\mathrm{Ker}(f_2) \longrightarrow \mathrm{Ker}(f_3)$. Since we assume that the reader has verified that we have a complex we need only to show that any element in the kernel of δ is the image of some element in $\mathrm{Ker}(f_2)$. Let x be an element of $\mathrm{Ker}(f_3)$ which is mapped to zero by δ, that is $x \in \ker(\delta)$. We need show that there exists an $x' \in \mathrm{Ker}(f_2)$ such that $\varphi_2(x') = x$. Let $x' \in M_2$ be any pre-image of x, and let $y \in N_1$ be the element such that $\psi_1(y) = f_2(x')$. Then we have that $\delta(x)$ is the image of y by the quotient map $N_1 \longrightarrow \mathrm{coker}(f_1)$. We have assumed that $\delta(x) = 0$, which then means that $y = f_1(z)$, for some $z \in M_1$. Now consider the element $x' - \varphi_1(z) \in M_2$, which is a pre-image of x, as

$$\varphi_2(x' - \varphi_1(z)) = \varphi_2(x') - \varphi_2(\varphi_1(z)) = x - 0.$$

Using the commutativity of the left diagram we obtain that

$$f_2(x' - \varphi_1(z)) = f_2(x') - f_2(\varphi_1(z)) = f_2(x') - \psi(f_1(z)).$$

As $f_1(z) = y$ and $\psi_1(y) = f_2(x')$, we get that $f_2(x' - \varphi_1(z)) = 0$. We therefore have an element $x' - \varphi_1(z)$ in the kernel of f_2 which is mapped to the element x. Consequently $\mathrm{Ker}(\delta)$ equals the image of $\mathrm{Ker}(f_2)$. □

2.3 Exercises

2.3.1. Let N_1 and N_2 be two submodules of M. Show that we have a short exact sequence

$$0 \longrightarrow N_1 \cap N_2 \longrightarrow N_1 \oplus N_2 \longrightarrow N_1 + N_2 \longrightarrow 0,$$

where the first map sends x to (x, x), and the second map takes (x, y) to $x - y$.

2.3.2. Let $f \colon M \longrightarrow N$ and $g \colon N \longrightarrow P$ be two A-module homomorphisms. Show that there is an exact sequence

$$0 \to \mathrm{Ker}(f) \to \mathrm{Ker}(gf) \to \mathrm{Ker}(g) \to \mathrm{Coker}(f) \to \mathrm{Coker}(gf) \to \mathrm{Coker}(g) \to 0.$$

2.3.3. Given a short exact sequence $0 \to M \to N \to P \to 0$, and given $f \colon P' \to P$. Show that there exist a commutative diagram and exact rows of the form

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & M \arrow[r] & N \arrow[r] & P \arrow[r] & 0
\end{tikzcd}$$

2.3.4. Given chain maps $f \colon P_\bullet \longrightarrow Q_\bullet$ and $q \colon Q_\bullet \longrightarrow R_\bullet$ that, for every n, form the commutative diagrams and exact sequences of A-modules

$$\begin{tikzcd}
0 \arrow[r] & P_n \arrow[r, "f_n"] \arrow[d, "d_n^P"] & Q_n \arrow[r, "g_n"] \arrow[d, "d_n^Q"] & R_n \arrow[r] \arrow[d, "d_n^R"] & 0 \\
0 \arrow[r] & P_{n-1} \arrow[r, "f_{n-1}"] & Q_{n-1} \arrow[r, "g_{n-1}"] & R_{n-1} \arrow[r] & 0.
\end{tikzcd}$$

Show that we obtain induced commutative diagrams and exact sequences

$$\begin{tikzcd}
 & \mathrm{Coker}(d_{n+1}^P) \arrow[r] \arrow[d] & \mathrm{Coker}(d_{n+1}^Q) \arrow[r] \arrow[d] & \mathrm{Coker}(d_{n+1}^R) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(d_{n-1}^P) \arrow[r] & \mathrm{Ker}(d_{n-1}^Q) \arrow[r] & \mathrm{Ker}(d_{n-1}^R) &
\end{tikzcd}$$

2.3.5. Let

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0$$

be an exact sequence of finitely generated free A-modules F_i. Show that

$$\sum_{i=0}^{n} (-1)^i \mathrm{rank}(F_i) = 0.$$

Chapter 3

Hom-functors

3.1 The Hom-functors

Let M be an A-module. We have, for any A-module N, that the set of A-module homomorphisms from M to N, $\mathrm{Hom}_A(M, N)$, form an A-module. Note that if $f\colon N_1 \longrightarrow N_2$ is an A-module homomorphism we get an induced A-module homomorphism

$$f_*\colon \mathrm{Hom}_A(M, N_1) \longrightarrow \mathrm{Hom}_A(M, N_2),$$

by composition; a homomorphism $\varphi\colon M \longrightarrow N_1$ is mapped to $f_*(\varphi) := f \circ \varphi$.

3.1.1 Proposition. *Let M be an A-module, and let*

$$0 \longrightarrow N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3 \longrightarrow 0$$

be a short exact sequence of A-modules. Then we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M, N_1) \xrightarrow{f_*} \mathrm{Hom}_A(M, N_2) \xrightarrow{g_*} \mathrm{Hom}_A(M, N_3).$$

Proof. Let $\varphi\colon M \longrightarrow N_1$ be an A-module homomorphism. We have that $g_* \circ f_*(\varphi)$ is the composition of the three homomorphisms g, f and φ. As the composition of g and f is the zero map, it follows that the sequence (3.1.1) is a complex. To establish injectivity of f_*, we assume that φ is in the kernel of f_*. That is, the composition

$$M \xrightarrow{\varphi} N_1 \xrightarrow{f} N_2$$

is the zero map. If $\varphi\colon M \longrightarrow N_1$ was not the zero map, there would exist an $x \in M$ such that $\varphi(x) \neq 0$. But then $f_*(\varphi)$ would take that particular element to $f(\varphi(x))$, which would be non-zero as f is injective. It follows that φ is the zero homomorphism, and f_* is injective. Now we want to show that $\mathrm{Im}(f_*) = \mathrm{Ker}(g_*)$. As we already have seen that the sequence (3.1.1) is a complex, we need only to show that $\mathrm{Ker}(g_*) \subseteq \mathrm{Im}(f_*)$. Let $\varphi\colon M \longrightarrow N_2$ be an element of the kernel of g_*. Then in particular we have $g_*(\varphi(x)) = 0$ for every $x \in M$. As the image of f equals the kernel of g, we obtain that the image of φ is contained in the image of f, that is $\mathrm{Im}(\varphi) \subseteq \mathrm{Im}(f)$. Furthermore, as f is injective each element in the image has an unique preimage in the domain. Define now the map of sets $\varphi'\colon M \longrightarrow N_1$ by taking $x \in M$ to the unique preimage of $\varphi(x)$. This is not only a well-defined map, it is readily checked that in fact this is an A-module homomorphism. By construction we have that $f_*(\varphi') = \varphi$, and consequently $\ker(g_*) = \mathrm{Im}(f_*)$, and we have proven the proposition. $\square$

3.1.2 Example. Our favorite example is the short exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{2} \mathbf{Z} \longrightarrow \mathbf{Z}/(2) \longrightarrow 0.$$

Apply now the functor $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/(2), -)$ to that sequence and we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/(2)), \mathbf{Z}) \xrightarrow{2_*} \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/(2), \mathbf{Z}) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/(2), \mathbf{Z}/(2)).$$

By Exercise (1.8.23) we have that the module $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/(2), \mathbf{Z})$ is the zero module. On the other hand we clearly have that $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/(2), \mathbf{Z}/(2)) \neq 0$, as it contains the identity morphism. We have that the sequence considered above is exact, but we have just shown that the rightmost arrow is not surjective. The sequence in question is

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/(2), \mathbf{Z}/(2)) = \mathbf{Z}/(2).$$

3.2 Projective modules

The assignment taking an A-module N to the A-module $\mathrm{Hom}_A(M, N)$ is functorial. We say that $\mathrm{Hom}_A(M, -)$ is a *left exact* functor as the functor takes short exact sequences to left exact sequences. As the Example (3.1.2) above shows, the functor is not right exact, in general, since right exactness is not preserved. One could however ask for which modules M the functor $\mathrm{Hom}_A(M, -)$ takes short exact sequences to short exact sequences. Such modules certainly exist, take $M = 0$ for instance. A functor that takes short exact sequences to short exact sequences is called an *exact functor.*

3.2.1 Definition. *An A-module P is* projective *if for any surjection of A-modules $Q' \longrightarrow Q$, any homomorphism $g\colon P \longrightarrow Q$ can be extended to a homomorphism $g'\colon P \longrightarrow Q'$. That is, given the diagram of A-modules,*

$$\begin{tikzcd} & P \arrow[d, "g"] \arrow[dl, dashed] & \\ Q' \arrow[r] & Q \arrow[r] & 0, \end{tikzcd}$$

and solid arrows representing homomorphisms, with the horizontal sequence exact. Then there exists a homomorphism, represented by the dotted arrow, making the diagram commutative.

3.2.2 Example. Any free module satisfies the defining property of a projective module. Hence a free module is projective.

3.2.3 Theorem. *A module P is projective if and only if* $\mathrm{Hom}_A(P, -)$ *is an exact functor.*

Proof. Let P be a projective module. By (3.1.1) we have that $\mathrm{Hom}_A(P, -)$ is left-exact. We need only to check that it also preserves surjections. Let $f\colon N \longrightarrow N'$ be a surjective A-module homomorphism. We need to show that the associated A-module homomorphism

$$\mathrm{Hom}_A(P, N) \xrightarrow{f_*} \mathrm{Hom}_A(P, N') \tag{3.1}$$

is surjective. Let $\varphi \in \mathrm{Hom}_A(P, N')$, and consider the diagram

$$\begin{tikzcd} N \arrow[r, "f"] & N' \arrow[r] & 0\,. \\ & P \arrow[u, "\varphi"] & \end{tikzcd} \tag{3.2}$$

The horizontal sequence is exact, and by definition of projectivity there exists a lifting $\varphi_N\colon P \longrightarrow N$ of φ. Then we have that $f_*(\varphi_N) = \varphi$, and the homomorphism (3.1) is surjective. Conversely, assume that $\mathrm{Hom}_A(P, -)$ is an exact functor. In particular that means that for any surjection $f\colon N \longrightarrow N'$, and any diagram as (3.2), there would exist a lifting since the module homomorphism (3.1) is surjective. In other words P is projective. $\square$

3.3 Injective modules

We now consider the dual situation, fixing N instead of M when considering $\mathrm{Hom}_A(M,N)$. If $f\colon M_1 \longrightarrow M_2$ is an A-module homomorphism we get an induced A-module homomorphism

$$f^*\colon \mathrm{Hom}(M_2,N) \longrightarrow \mathrm{Hom}(M_1,N),$$

given by composition $\varphi \mapsto \varphi \circ f$. Note that the arrows get reversed, and we say that $\mathrm{Hom}_A(-,N)$ is a *contravariant*-functor.

3.3.1 Proposition. *Let N be an A-module, and let*

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

be a short exact sequence of A-modules. Then we have an induced exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M_3,N) \xrightarrow{g^*} \mathrm{Hom}_A(M_2,N) \xrightarrow{f^*} \mathrm{Hom}_A(M_1,N)$$

of A-modules.

Proof. That we have a complex follows from the fact that $g \circ f$ is the zero map. Let us first establish injectivity of g^*. Let $\varphi \in \mathrm{Hom}_A(M_3,N)$ be such that $g^*(\varphi)=0$. Thus $\varphi \circ g\colon M_2 \longrightarrow N$ is the zero map, which is equivalent with $\mathrm{Ker}(\varphi) \supseteq \mathrm{Im}(g)$. As the homomorphism $g\colon M_2 \longrightarrow M_3$ is surjective, it is necessarily so that $\mathrm{Ker}(\varphi)=M_3$. That is, φ is the zero map, and we have injectivity of g^*. To prove exactness in the middle we need to see that $\mathrm{Ker}(f^*) \subseteq \mathrm{Im}(g^*)$. Let $\varphi \in \mathrm{Hom}_A(M_2,N)$ be such that $f^*(\varphi)=0$. Then $\varphi\circ f$ is the zero map, which means that $\mathrm{Im}(f) \subseteq \mathrm{Ker}(\varphi)$. As any homomorphism factors through its kernel (Lemma 1.6.3, Assertion (3)), we have in particular that φ factors through

$$\overline{\varphi}\colon\ M_2/\mathrm{Im}(f) = M_3 \longrightarrow N.$$

Then we have that $g^*(\overline{\varphi})=\varphi$, hence φ is in the image of g^*. In other words $\mathrm{Hom}_A(-,N)$ is left exact. □

3.3.2 Example. Let us again consider our favorite example, the exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{2} \mathbf{Z} \longrightarrow \mathbf{Z}/(2) \longrightarrow 0.$$

Apply $\mathrm{Hom}_{\mathbf{Z}}(-, \mathbf{Z}/(2))$ to the sequence above, and we get the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/(2), \mathbf{Z}/(2)) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}, \mathbf{Z}/(2)) \xrightarrow{2^*} \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}, \mathbf{Z}/(2)).$$

Since $\mathbf{Z}$ is a free module over itself, a homomorphism $\mathbf{Z} \longrightarrow N$ is determined by its action on 1. Furthermore, as such a homomorphism can send 1 to anything in N we have that $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}, \mathbf{Z}/(2)) = \mathbf{Z}/(2)$. The identification is thus to send a homomorphism $f\colon \mathbf{Z} \longrightarrow \mathbf{Z}/(2)$ to $f(1) \in \mathbf{Z}/(2)$. Note that under that identification the morphism

$$2^*\colon \mathbf{Z}/(2) \longrightarrow \mathbf{Z}/(2)$$

corresponds to multiplication by 2. In other words 2^* is the zero homomorphism, and we have that the sequence above is not surjective at the right.

3.3.3 Definition. *An A-module I is* injective *if for any injective homomorphism $Q \longrightarrow Q'$, we have that any homomorphism $\varphi\colon Q \longrightarrow I$ can be extended to a homomorphism $\varphi'\colon Q' \longrightarrow I$. That is, given a diagram of A-modules,*

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & Q \arrow[u, "\varphi"] \arrow[r] & Q' \arrow[ul, dashed] \end{tikzcd}$$

and solid arrows representing homomorphisms, with the horizontal sequence exact, there exists a dotted arrow making the diagram commutative.

3.3.4 Proposition. *An A-module I is injective if and only if any A-module homomorphism $f\colon J \longrightarrow I$, with $J \subseteq A$ an ideal, can be extended to a homomorphism $f'\colon A \longrightarrow I$.*

Proof. As ideals $J \subseteq A$ are particular cases of modules we immediately have that if I is an injective module, then any homomorphism from an ideal $J \longrightarrow I$ can be extended to a homomorphism $A \longrightarrow I$. We need to prove the converse. Let I be an A-module such that any homomorphism from ideals in A can be extended to homomorphism from the ring. We shall show that I is injective. Assume that we are given a diagram

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & Q \arrow[u, "\varphi"] \arrow[r] & Q' \end{tikzcd}$$

of A-modules where the sequence is exact. Let $\mathscr{A}$ denote the collection of pairs $\{(P, \phi)\}$ where P is an A-module such that $Q \subseteq P \subseteq Q'$, and where $\phi\colon P \longrightarrow I$ is an extension of φ. By Zorn's Lemma (see Exercise (3.4.12)) there exists a maximal element (P', ϕ') of $\mathscr{A}$. Note that if $P' = Q'$ then we get that I is injective, completing our proof. We therefore only need to show that $P' = Q'$. Assume therefore that $P' \neq Q$, and in particular there exists an element $x \in Q'$ but where $x \notin P'$. Define the ideal

$$J = \{a \in A \mid ax \in P'\} \subseteq A,$$

and the A-module homomorphism $f\colon J \longrightarrow I$ by sending a to $\phi'(ax)$. By assumption there exists an extension $f'\colon A \longrightarrow I$ of f, and we will use this extension to define an element (P'', ϕ'') of $\mathscr{A}$. Namely, let P'' denote the A-module generated by elements in P' and by x. Elements in P'' are then of the form $p + ax$, with $p \in P$ and $a \in A$. Furthermore, P'' is contained in Q', and contains P' as a proper submodule since $x \notin P'$. Define the A-module homomorphism

$$\phi''\colon P'' \longrightarrow I$$

by sending $p+ax$ to $\phi'(p)+a\cdot f'(1)$. We have by Exercise (3.4.13) that ϕ'' is a well-defined A-module homomorphism. Given that, we have now constructed a pair (P'', ϕ'') in $\mathscr{A}$ that violates the maximality of the pair (P', ϕ'). Hence our assumption that $P' \neq Q'$ fails and we have indeed that $P' = Q'$. □

3.3.5 Proposition. *An A-module I is injective if and only if the contravariant functor $\mathrm{Hom}_A(-, I)$ is exact.*

Proof. By the left exactness (3.3.1) we need only to see that the contravariant functor $\mathrm{Hom}_A(-, I)$ is right exact. Let $f\colon Q \longrightarrow Q'$ be an injective A-module homomorphism, and consider the associated A-module homomorphism

$$\mathrm{Hom}_A(Q', I) \xrightarrow{f^*} \mathrm{Hom}_A(Q, I). \tag{3.3}$$

Let $\varphi \in \mathrm{Hom}_A(Q, I)$ be an element. We need to see that there exists an element $\varphi' \in \mathrm{Hom}_A(Q', I)$ such that $f^*(\varphi') = \varphi$. However, we have that $\varphi\colon Q \longrightarrow I$ is an A-module homomorphism, and that $Q \longrightarrow Q'$ is injective. By definition there exists an extension $\varphi'\colon Q' \longrightarrow I$ of φ. That is $\varphi = \varphi' \circ f = f^*(\varphi')$, and consequently (3.3) is surjective.

Conversely, assume that $\mathrm{Hom}_A(-, I)$ is exact functor. Then in particular we have that for any injective homomorphism $f\colon Q \longrightarrow Q'$ the associated homomorphism (3.3) is surjective. Written out, that translates to the fact that to any homomorphism $\varphi\colon Q \longrightarrow I$ there exists a homomorphism $\varphi'\colon Q' \longrightarrow I$ such that $\varphi = \varphi' \circ f$. By definition, that means that I is injective. □

Divisible modules

The injective modules are much harder to write down than the projective ones; there are however plenty of injective modules. In fact any A-module M can be embedded into an injective module. (A proof of that fact is found in most text books on homological algebra). The injective $\mathbf{Z}$-modules are however relatively easy to describe.

3.3.6 Definition. *A $\mathbf{Z}$-module M is* divisible *if for any element $x \in M$, and any non-zero integer n there exist $y \in M$ such that $ny = x$.*

3.3.7 Example. An example of a divisible module is the field of rationals $\mathbf{Q}$.

3.3.8 Proposition. *A $\mathbf{Z}$-module I is injective if and only if it is divisible.*

Proof. Assume that I is an injective module. Let $x \in I$ be a non-zero element, and let $n > 0$ be a positive integer. We need to show that there exist $q \in I$ such that $nq = x$. Let $Q = \mathbf{Z}$ and let $\varphi\colon Q \longrightarrow I$ be the $\mathbf{Z}$-module homomorphism that sends $m \mapsto mx$, and let $\mu_n\colon Q \longrightarrow \mathbf{Z} = Q'$ be the multiplication map by n. The multiplication map is injective, and by definition of I being injective there exist an extension $\varphi'\colon \mathbf{Z} \longrightarrow I$ of φ. In other words $\varphi' \circ \mu_n = \varphi$. Set $q = \varphi'(1)$. We have

$$nq = n\varphi'(1) = \varphi'(n) = \varphi'(\mu_n(1)) = \varphi(1) = x.$$

So, to each non-zero element $x \in I$, and every non-zero integer n, we have shown there exist an element $q \in I$ such that $nq = x$. Thus the injective module I is divisible.

Conversely, assume that I is a divisible module. To see that I is injective we need to see, by Proposition (3.3.4), that any homomorphism from an ideal $J \subseteq \mathbf{Z}$ to I extends to $\mathbf{Z}$. Let $f\colon J \longrightarrow I$ be an A-module homomorphism, with J an ideal. Since any ideal in $\mathbf{Z}$ is principally generated, there exists an $m \in \mathbf{Z}$ such that $(m) = J$. Let $x = f(m)$. Since I is assumed injective there exists an $y \in I$ such that $my = x$. Let $f'\colon \mathbf{Z} \longrightarrow I$ be the homomorphism determined by sending 1 to y. We then have our extension of $f\colon J \longrightarrow I$, and I is injective. □

3.4 Exercises

3.4.1. Show that a free module is projective.

3.4.2. Let $M = \oplus_{i=1}^{n} M_i$. Show that M is projective if and only if each summand M_i is projective.

3.4.3. Let $A = \mathbf{Z}/(6)$, and consider the modules given by the ideals $M = (2) \subset A$ and $N = (3) \subset A$. Show that $M \oplus N$ is projective, but not free.

3.4.4. Let $I \subseteq A$ be an ideal. Show that for any A-module M the A-module $\mathrm{Hom}_A(A/I, M)$ is naturally an A/I-module. Show furthermore that $\mathrm{Hom}_A(A/I, -)$ takes injective A-modules to injective A/I-modules.

3.4.5. Let P be a projective A-module, and let $I \subseteq A$ be an ideal. Show that P/IP is a projective A/I-module.

3.4.6. Let P be a projective A-module that can be generated by n elements. Show that the dual module $\mathrm{Hom}_A(P, A)$ is also projective and can be generated by n elements.

3.4.7. Given short exact sequences $0 \to M \to P \to N \to 0$ and $0 \to M' \to P' \to N \to 0$, where the modules P and P' are assumed to be projective. Show that $M \oplus P' = M' \oplus P$.

3.4.8. Let m and n be two coprime integers. Show that $\mathbf{Z}/(m)$ is a projective $\mathbf{Z}/(mn)$-module, but not a free module.

3.4.9. Show that $\mathbf{Q}$ is not a projective $\mathbf{Z}$-module.

3.4.10. Are submodules of projective modules projective? Consider $(p) \subset A = \mathbf{Z}/(p^2)$.

3.4.11. Let $f\colon M \longrightarrow N$ be an A-module homomorphism of free A-modules. Show that the following are equivalent.

(1) We have $f(M)$ as a direct summand of N.

(2) We have that $\mathrm{Coker}(f)$ is projective.

(3) There exists an A-module homomorphism $\varphi\colon N \longrightarrow M$ such that $f = f\varphi f$.

3.4.12. Let $\mathscr{A}$ be a non-empty set. A subset $\mathscr{R} \subseteq \mathscr{A} \times \mathscr{A}$ is a relation on $\mathscr{A}$. Assume that the relation $\mathscr{R}$ satisfies the following three axioms.

1. Reflexitivity: For any $x \in \mathscr{A}$ we have $(x, x) \in \mathscr{R}$.
2. Antisymmetry: If $(x, y) \in \mathscr{R}$ and $(y, x) \in \mathscr{R}$, then $x = y$.
3. Transitivity: If $(x, y) \in \mathscr{R}$ and $(y, z) \in \mathscr{R}$, then $(x, z) \in \mathscr{R}$.

Then we say that $\mathscr{A}$ is *partially ordered*. A subset $\mathscr{D}$ of a partially ordered set $\mathscr{A}$ is a *chain* if for any pair $(x, y) \in \mathscr{D} \times \mathscr{D}$ we have either $(x, y) \in \mathscr{R}$ or $(y, x) \in \mathscr{R}$. A subset $\mathscr{D}$ has an *upper bound* in $\mathscr{A}$ if there exist $w \in \mathscr{A}$ such that $(x, w) \in \mathscr{R}$, for every $x \in \mathscr{D}$.

The Zorn's Lemma states that if every chain $\mathscr{D}$ of a non-empty partially ordered set $\mathscr{A}$ has an upper bound in $\mathscr{A}$, then $\mathscr{A}$ has at least one maximal element. Show that the Zorn Lemma applies to the collection of extensions considered in the proof of Proposition (3.3.4). The Zorn's lemma itself is equivalent with the axiom of choice, for instance, and simply taken as a fact in these notes.

3.4.13. Let $P \subseteq Q$ be A-modules, and assume that there exists an $x \in Q \setminus P$. Let $P_x \subseteq Q$ denote the submodule of elements of the form $p + ax$, with $p \in P$, and $a \in A$. Show that there is, for any A-module M, a well-defined A-module homomorphism

$$\mathrm{Hom}_A(P, M) \oplus \mathrm{Hom}_A(A, M) \longrightarrow \mathrm{Hom}_A(P_x, M)$$

sending (φ, f) to the homomorphism $\phi\colon P_x \longrightarrow M$, where

$$\phi(p + ax) = \varphi(p) + f(a).$$

Chapter 4

Homology

4.1 Homology

Let $P_\bullet = \{P_n, d_n\}$ be a complex (with descending boundary operators). We define the A-module of *n-cycles* as $Z_n(P) = \ker(d_n)$, and the A-module of *n-boundaries* as $B_n(P) = \mathrm{im}(d_{n+1})$.
Since the collection of modules form a complex we have $B_n(P) \subseteq Z_n(P)$. We define the n'th *homomology module* of the complex
$\{P_n, d_n\}$ as the A-module

$$H_n(P) := Z_n(P)/B_n(P).$$

Note that the complex $\{P_n, d_n\}$ is exact if and only if all of its associated homology modules $H_n(P) = 0$, for all n.

4.1.1 Example. Consider the complex $P_\bullet$ we introduced in Example (2.1.3);

$$0 \longrightarrow \mathbf{Z} \xrightarrow{d_1=4} \mathbf{Z} \xrightarrow{d_0} \mathbf{Z}/(2) \longrightarrow 0$$

Only the zero degree homomology $H_0(P)$ is of interest, the other homomology modules are zero. We have

$$H_0(P) = \ker(d_0)/\mathrm{Im}(4) = (2)/(4) \backsimeq \mathbf{Z}/(2).$$

4.1.2 Proposition. *Let $f\colon P_\bullet \longrightarrow Q_\bullet$ be a chain map of complexes. Then we have an induced A-module homomorphism of*
n-cycles $Z_n(P) \longrightarrow Z_n(Q)$ and n-boundaries $B_n(P) \longrightarrow B_n(Q)$. In particular we get an induced A-module homomorphism of n'th homology modules

$$f_{n,*}\colon H_n(P) \longrightarrow H_n(Q) \quad \textit{for all} \quad n.$$

Proof. The two first statements follow from the commutative diagram

$$\begin{tikzcd} P_n \arrow[r, "f_n"] \arrow[d, "d_n^P"] & Q_n \arrow[d, "d_n^Q"] \\ P_{n-1} \arrow[r, "f_{n-1}"'] & Q_{n-1} \end{tikzcd} .$$

Indeed, let $x \in Z_n(P)$. Then $d_n^P(x) = 0$. As

$$0 = f_{n-1}(d_n^P(x)) = d_n^Q(f_n(x)),$$

we have $f_n(x) \in \ker(d_n^Q)$. That is $f_n(x) \in Z_n(Q)$. Similarly with boundaries; let $x \in B_{n-1}(P)$. Then there exist $y \in P_n$ such that $x = d_n^P(y)$. And then $f_{n-1}(x) = d_n^Q(f_n(y))$, so $f_{n-1}(x) \in B_n(Q)$. The last assertion in the lemma is a consequence of the first two.

□

Exact triangles

Assume that we have two chain maps of complexes $f\colon P_\bullet \longrightarrow Q_\bullet$ and $g\colon Q_\bullet \longrightarrow R_\bullet$ such that
on each degree n we have the short exact sequence

$$0 \longrightarrow P_n \xrightarrow{f_n} Q_n \xrightarrow{g_n} R_n \longrightarrow 0.$$

We say then that we have a short exact sequence of complexes.

4.1.3 Proposition (Exact triangle). *Let $f\colon P_\bullet \longrightarrow Q_\bullet$ and $g\colon Q_\bullet \longrightarrow R_\bullet$ be two chain maps that composed give a short exact sequence of complexes. Then we get induced a long* exact *sequence of homomology modules*

$$\cdots \longrightarrow H_n(P) \xrightarrow{f_{n,*}} H_n(Q) \xrightarrow{g_{n,*}} H_n(R) \xrightarrow{\delta_{n,*}} H_{n-1}(P) \longrightarrow \cdots .$$

Proof. We note that any homomorphism $d_n^P\colon P_n \longrightarrow P_{n-1}$ will factor through its image $B_n(P)$. We therefore have commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & P_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d^P_{n+1}"] & Q_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d^Q_{n+1}"] & R_{n+1} \arrow[r] \arrow[d, "d^R_{n+1}"] & 0\ . \\ 0 \arrow[r] & B_n(P) \arrow[r, "f_n"] & B_n(Q) \arrow[r, "g_n"] & B_n(R) \arrow[r] & 0 \end{tikzcd} \tag{4.1}$$

The upper row is exact by assumption, and the reader can check that the bottom row is also exact. By Exercise (2.3.4) we get the
commutative diagram

$$\begin{tikzcd} & P_n/B_n(P) \arrow[r] \arrow[d] & Q_n/B_n(Q) \arrow[r] \arrow[d] & R_n/B_n(R) \arrow[r] \arrow[d] & 0\ , \\ 0 \arrow[r] & Z_{n-1}(P) \arrow[r] & Z_{n-1}(Q) \arrow[r] & Z_{n-1}(R) & \end{tikzcd}$$

where the horizontal sequences are exact. Now, the cokernel of each of the vertical maps are the homomology modules of degree $n-1$, and the kernel of
the vertical maps are the homology modules of degree n. The Snake Lemma (2.2.3) applied to the last diagram proves the proposition.

□

4.1.4 Remark. The name exact triangle comes from the mnemonic triangle

$$\begin{tikzcd} H(P) \arrow[rr, "f_*"] & & H(Q)\ . \arrow[dl, "g_*"] \\ & H(R) \arrow[ul, "\delta_*"] & \end{tikzcd}$$

4.1.5 Example. Consider the the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbf{Z} \arrow[r, "1"] \arrow[d, "4"] & \mathbf{Z} \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & \mathbf{Z} \arrow[r, "2"] & \mathbf{Z} \arrow[r] & \mathbf{Z} \arrow[r] & 0 \end{tikzcd}$$

The horizontal two sequences are exact, and we consider the three middle vertical terms as complexes $P_\bullet, Q_\bullet$ and $R_\bullet$, read from the left to right.

The degrees of the complexes shown are the degree zero and the degree one part. The diagram then shows that we have chain maps $f\colon P_\bullet \longrightarrow Q_\bullet$ and $g\colon Q_\bullet \longrightarrow R_\bullet$ that composed give a long exact sequence in homology. The two boundary maps d_1^P and d_1^Q are injective, hence $H_1(P) = H_1(Q) = 0$. The long exact sequence in homology becomes

$$0 \longrightarrow H_1(R) = \mathbf{Z} \xrightarrow{d_{1,*}} H_0(P) = \mathbf{Z} \xrightarrow{2_*} H_0(Q) = \mathbf{Z}/(4) \longrightarrow 0.$$

Since the sequence is exact it follows that $d_{1,*}$ is the homomorphism given by multiplication by two.

4.2 Homotopy

A chain map $f\colon P_\bullet \longrightarrow Q_\bullet$ is *null-homotopic* if there exist A-module homomorphisms $s_n\colon P_n \longrightarrow Q_{n+1}$ such that

$$f_n = d_{n+1}^Q \circ s_n + s_{n-1} \circ d_n^P,$$

for every n. Two chain maps f and g from a complex $P_\bullet$ to another complex $Q_\bullet$ are *homotopic* if the chain map $f - g\colon P_\bullet \longrightarrow Q_\bullet$ is null-homotopic.

4.2.1 Remark. Here we have given the definition of homotopy for a complex where the boundary operators are descending $d_n\colon P_n \longrightarrow P_{n-1}$. If the complex had ascending boundary operators then the definition of homotopy will be essentially the same, however the A-module homomorphisms s_n would then be descending the degree $s_n\colon P_n \longrightarrow Q_{n-1}$. The equation these homomorphisms should satisfy in order for a chain map to be null-homotopic would then read

$$f_n = d_{n-1}^Q s_n + s_{n+1} d_n^P.$$

4.2.2 Example. Consider the two complexes drawn below as horizontal sequences, showing degrees $\{2, 1, 0, -1\}$,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbf{Z} \arrow[r] \arrow[d, "2"] & \mathbf{Z}/(8) \arrow[r] \arrow[d] & 0 \\
\mathbf{Z}/(8) \arrow[r] & \mathbf{Z}/(4) \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

If $P_\bullet$ is the upper complex, and $Q_\bullet$ is the lower, we have in the diagram shown a chain map $f\colon P_\bullet \longrightarrow Q_\bullet$. We claim that the chain map f is null-homotopic. Define the $\mathbf{Z}$-module homomorphisms

$$s_1\colon \mathbf{Z} = P_1 \longrightarrow Q_2 = \mathbf{Z}/(8) \quad s_0\colon \mathbf{Z}/(8) = P_0 \longrightarrow \mathbf{Z}/(4) = Q_1$$

as the canonical projections. All other s_n we set to be the zero map. We then have $f_n = d^Q_{n+1}s_n + s_{n-1}d^P_n$, for all n, and we have that the chain map f is null-homotopic.

4.2.3 Theorem. *Two chain maps between two complexes $P_\bullet$ and $Q_\bullet$ that are homotopic, induce the same homomorphism of homology modules. That is, if f and g are homotopic chain maps then*

$$f_{n*} = g_{n*} \colon H_n(P) \longrightarrow H_n(Q),$$

for all n.

Proof. We need to show that a null-homotopic map $F = f - g$ is the zero map. Let $x \in H_n(P)$ be an element in the n-th homomology. Recall that $H_n(P)$ is the quotient module $Z_n(P)/B_n(P)$, and therefore we may represent x with an n-cycle $z \in Z_n(P)$. We have that the null-homotopy F sends the n-cycle to

$$F(z) = d^Q_{n+1}s_n(z) + s_{n-1}d^P_n(z).$$

Since z is an n-cycle we have by definition that $d^P_n(z) = 0$. Thus $F(z) = d^Q_{n+1}s_n(z)$. As $s_n(z) \in Q_{n+1}$ we have by definition that $d^Q_{n+1}(s_n(z))$ is an n-boundary, in other words $F(z) \in B_n(Q)$. Then, by definition of the homology, we have that $F(z) = 0$ in $H_n(Q)$, and we have proven our claim.

□

4.3 Exercises

4.3.1. Let $f\colon P_\bullet \longrightarrow Q_\bullet$ be a chain map of complexes. We define the *mapping cone* in the following way. Let $M_n = P_{n-1} \oplus Q_n$, and define $d^M_n \colon M_n \longrightarrow M_{n-1}$ by

$$d^M_n(x, y) = (-d^P_{n-1}(x), d^Q_n(y) + f(x)).$$

Show that (M_n, d^M_n) forms a complex, and that we have a long exact sequence

$$\cdots \longrightarrow H_n(Q) \longrightarrow H_n(M) \longrightarrow H_{n-1}(P) \longrightarrow \cdots.$$

4.3.2. Let $f\colon P_\bullet \longrightarrow Q_\bullet$ be a chain map. Show that, for every n, the composite map

$$P_{n+1} \xrightarrow{d^P_{n+1}} P_n \xrightarrow{f_n} Q_n$$

will factor through the boundary $B_n(Q) \subseteq Q_n$.

4.3.3. Let f and g be two homotopic chain maps $P_\bullet \longrightarrow Q_\bullet$, and let $s_n \colon P_n \longrightarrow Q_{n+1}$ be the A-module homorphisms such that $g_n - f_n = s_{n-1}d_n^Q + d_{n+1}^P s_n$, for all n. Define the A-module homomorphism $F \colon P_n \longrightarrow Q_n$ by $F = g_n - f_n - s_{n-1}d_n^P$. Show that $\mathrm{Im}(F) \subseteq \mathrm{Im}(d_{n+1}^Q)$.

4.3.4. Given two short exact sequences of complexes, and commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & P'_\bullet \arrow[r] \arrow[d, "f'"] & P_\bullet \arrow[r] \arrow[d, "f"] & P''_\bullet \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & Q'_\bullet \arrow[r] & Q_\bullet \arrow[r] & Q''_\bullet \arrow[r] & 0.
\end{tikzcd}$$

Show that we have an induced chain map from the long exact sequence in homology. That is, that we have commutative diagrams

$$\begin{tikzcd}
\arrow[r] & H_n(P') \arrow[r] \arrow[d, "{f'}_*"] & H_n(P) \arrow[r] \arrow[d, "f_*"] & H_n(P'') \arrow[r, "\delta"] \arrow[d, "{f''}_*"] & H_{n-1}(P') \arrow[r] \arrow[d, "f'_*"] & {} . \\
\arrow[r] & H_n(Q') \arrow[r] & H_n(Q) \arrow[r] & H_n(Q'') \arrow[r, "\delta"] & H_{n-1}(Q') \arrow[r] & {}
\end{tikzcd}$$

Chapter 5

Resolutions and Ext

5.1 Free resolutions

Let M be an A-module. Assume that we have an exact sequence

$$\cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow F_0 \xrightarrow{d_0} M \longrightarrow 0.$$

Then the complex $\{F_n, d_n\}$ ($n \geq 0$, without M) is said to be a *resolution* of M. If, furthermore, the A-modules F_n all are free, then we say that $\{F_n, d_n\}$ is a *free resolution* of M.

5.1.1 Remark. Note that if $\{F_n, d_n\}_{n\geq 0}$ is a resolution of M then the module M itself is not a part of the data. However, we recover M as the cokernel $M = \operatorname{coker}(d_1)$. Finally, as the sequence ends at F_0, there is strictly speaking no boundary operator in degree zero. When we occasionally anyhow refer to d_0 of a resolution, we simply mean the projection map $F_0 \longrightarrow \operatorname{coker}(d_1) = M$.

5.1.2 Proposition. *Any module over any ring admits a free resolution.*

Proof. Let M be an A-module. By Proposition (1.6.5) we can write M as a quotient module of a free A-module F_0. As the kernel K_0 of the quotient homomorphism $F_0 \longrightarrow M$ again is an A-module, we can write K_0 as a quotient of a free A-module F_1. This process is clearly inductive, giving us free modules F_{n+1} that surjects down to the kernel $F_n \longrightarrow K_{n-1}$. The quotient homomorphism $F_n \longrightarrow K_{n-1}$ composed with the injection $K_{n-1} \longrightarrow F_{n-1}$ give a sequence of A-modules and homomorphisms

$$\cdots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0,$$

which is a resolution of M. $\square$

5.2 Finite resolutions

Let $\{F_n, d_n\}$ be a free resolution of an A-module M. We say that $\{F_n, d_n\}$ is a *finite free* resolution if the following two conditions are satisfied.

(1) All the modules F_n are free of finite rank.

(2) Only a finite number of the modules F_n are non-zero modules, that is $F_n = 0$ for $n >> 0$.

An A-module M admits a finite free resolution if there exists at least one finite free resolution of M. Furthermore, if $\{F_n, d_n\}$ is a finite resolution of M then the smallest integer p such that $F_n = 0$ for all $n > p$, is called the *length* of the resolution.

5.2.1 Example. A finite dimensional vector space V is free considered as a module over the base field. In particular any finite dimensional vector space has trivially a finite free resolution. Thus, for a finite dimensional vector space we can always find a finite free resolution of length zero.

5.2.2 Proposition. *Let $A = \mathbf{Z}$ be the ring of integers. Any submodule F of a finitely generated free module G, is free of finite rank.*

Proof. A submodule of a finitely generated $\mathbf{Z}$-module is again finitely generated, see Exercise (1.8.29). Thus F is finitely generated, and as such it is by the structure Theorem for finitely generated abelian groups a direct sum of a free part and a torsion part. By being a submodule of a free module, it is clear that F has no torsion. Hence F is free. □

5.2.3 Corollary. *If M is a finitely generated $\mathbf{Z}$-module then M admits a finite free resolution of length at most one.*

Proof. Let M be a finitely generated module over $\mathbf{Z}$. By Proposition (1.6.5) we can write M as a quotient of a free module $F_0 = \oplus_{i=1}^{n}\mathbf{Z}$ of finite rank. Let F_1 denote the kernel of the quotient map, and we have an exact sequence of $\mathbf{Z}$-modules

$$0 \longrightarrow F_1 \longrightarrow \oplus_{i=1}^{n}\mathbf{Z} \longrightarrow M \longrightarrow 0.$$

By the proposition we have that F_1 is free and of finite rank. Consequently the sequence above is a finite free resolution. □

5.2.4 Remark. The previous example indicates how a free resolution is to be constructed in the general case, with M a finitely generated A-module. However, there are some complications for *both* conditions needed for a finite free resolution. By being finitely generated we have that M can be written as a quotient of a free A-module of finite rank F_0. Let K_0 denote the kernel, and consider the short exact sequence of A-modules

$$0 \longrightarrow K_0 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

In general K_0 would not be free, or not even finitely generated. However, if A is a *Noetherian ring* then any submodule of a finitely generated module would again be finitely generated. Fields and the ring of integers $\mathbf{Z}$ are noetherian, as well as polynomial rings in finite number of variables over a field, and their quotients. Given now that A were Noetherian, then K_0 would be finitely generated. Hence we could write K_0 as a quotient of some free finite rank A-module F_1. Let K_1 denote the kernel of $F_1 \longrightarrow K_0$, and we have a exact sequence

$$0 \longrightarrow K_1 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

and we have started constructing a finite free resolution of M. The other assumption, that the resolution is of finite length is not automatically satisfied for Noetherian rings in general. The question whether this process terminates after a finite steps is related to certain geometric properties of the ring A.

Finite and in particular free resolutions are suitable for computational aspects as it all become a matter of linear algebra techniques.

5.3 Projective resolutions

A projective resolution of an A-module M is a resolution

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_0 \xrightarrow{d_0} M \longrightarrow 0,$$

with projective modules P_n, for all $n \geq 0$.

5.3.1 Proposition. *Any module over any ring admits a projective resolution.*

Proof. We have seen that a free module is projective (Example 3.2.2). Hence, by Proposition (5.1.2) we have that any module over any ring admits a projective resolution. $\square$

5.3.2 Theorem. *Let $P_\bullet$ and $Q_\bullet$ be two resolutions of a module M, with $P_\bullet$ a projective resolution. Then there exists a chain map $f\colon P_\bullet \longrightarrow Q_\bullet$. Furthermore,* any *other chain map $g\colon P_\bullet \longrightarrow Q_\bullet$, inducing the identity on M, is homotopic to f.*

Proof. We first exhibit a chain map $f\colon P_\bullet \longrightarrow Q_\bullet$. By assumption the boundary map $d_n^Q\colon Q_0 \longrightarrow M$ is surjective, and it follows from projectivity assumption of P_0 that the boundary map $d_0^P\colon P_0 \longrightarrow M$ extends to a homomorphism $f_0\colon P_0 \longrightarrow Q_0$. We can iterate this in the following way. Assume that we have constructed homeomorphisms $f_{n'}\colon P_{n'} \longrightarrow Q_{n'}$, commuting with the boundary maps, for all integers between $0 \leq n' \leq n$, for some integer n. We will show how this yields a homomorphism $f_{n+1}\colon P_{n+1} \longrightarrow Q_{n+1}$. Let

$$B_n(Q) = \mathrm{im}(d_{n+1}^Q) \subseteq Q_n$$

denote the boundary. By Exercise (4.3.2) the composite

$$P_{n+1} \xrightarrow{d_{n+1}^P} P_n \xrightarrow{f_n} Q_n, \tag{5.1}$$

will factor through $B_n(Q) \subseteq Q_n$. As the boundary map $d_{n+1}^Q\colon Q_{n+1} \longrightarrow Q_n$ surjects down to $B_n(Q)$ we get by projectivity of P_n that the homomorphism (5.1) has an extension $f_{n+1}\colon P_n \longrightarrow Q_n$. We thereby obtain a chain map $f\colon P_\bullet \longrightarrow Q_\bullet$ by the projectivity of P_n.

Next we want to show that the chain map f is unique up to homotopy. Let $g\colon P_\bullet \longrightarrow Q_\bullet$ be another chain map, such that the induced map on M is the identity. Let $F_n = g_n - f_n$. By assumption $F_0 = g_0 - f_0$ induces the zero on M, so$F_0\colon P_0 \longrightarrow Q_0$ will factorize through the image $B_0(Q)$ of $d_1^Q\colon Q_1 \longrightarrow Q_0$. By the projectivity assumption of P_0 we now get a lifting $s_0\colon P_0 \longrightarrow Q_1$ of F_0. Consider now

$$g_1 - f_1 - s_0 d_1^P\colon P_1 \longrightarrow Q_1. \tag{5.2}$$

One checks that the composition of that particular map (5.2) with d_1^Q is zero, hence the map (5.2) factors through the image of $d_2^Q\colon Q_2 \longrightarrow Q_1$. Thus, by the projectivity assumption on P_1 there exists a lifting $s_1\colon P_1 \longrightarrow Q_2$ of (5.2). By induction we get A-module homomorphisms $s_n\colon P_n \longrightarrow Q_{n+1}$ for all $n \geq 0$, and by construction we have that

$$g_n - f_n = s_{n-1} d_n^P + d_n^Q s_n.$$

Hence $F = g - f$ is null-homotopic, and f is homotopic to g. $\square$

5.3.3 Definition. *Let M be an A-module and let $\{P_n, d_n\}$ be a projective resolution. For any A-module N we have the associated complex of A-modules*

$$0 \longrightarrow \mathrm{Hom}_A(P_0, N) \xrightarrow{d_1^*} \mathrm{Hom}_A(P_1, N) \xrightarrow{d_2^*} \cdots$$

The homology groups of the associated complex we denote by

$$\mathrm{Ext}^n_A(M, N) := \ker(d^*_{n+1})/\mathrm{im}(d^*_n),$$

for each $n \geq 0$.

5.3.4 Remark. Note that $\mathrm{Ext}^0_A(M, N) = \mathrm{Hom}_A(M, N)$, see Exercise (5.5.5).

5.3.5 Proposition. *The homology groups $\mathrm{Ext}^n_A(M, N)$ are independent of the choice of projective resolution.*

Proof. Let $P_\bullet$ and $Q_\bullet$ be two projective resolutions of M. By the Comparison Theorem (5.3.2) there exists a chain map $f\colon P_\bullet \longrightarrow Q_\bullet$ and a chain map $g\colon Q_\bullet \longrightarrow P_\bullet$. The composition gf is a chain map $P_\bullet \longrightarrow P_\bullet$. Then, by the Comparison Theorem again, we have that the chain map gf must be homotopic to the identity map $1\colon P_\bullet \longrightarrow P_\bullet$. In other words $1 - gf$ is null-homotopic. That is, there exist A-module homomorphisms $s_n\colon P_n \longrightarrow P_{n+1}$ such that

$$1 - gf = d_{n+1} \circ s_n + s_{n-1} \circ d_n.$$

When we apply $\mathrm{Hom}_A(-, N)$ to the resolutions $P_\bullet$ and $Q_\bullet$ we get

$$1^* - (gf)^* = (d_{n+1}s_n)^* + (s_{n-1}d_n)^*.$$

We have that $(gf)^* = f^* \circ g^*$, for any composition of A-module homomorphisms. Thus the identity above reads

$$1 - (gf)^* = s^*_n \circ d^*_{n+1} + d^*_n \circ s^*_{n-1}.$$

This is the equation for null-homotopy for an *ascending* chain complex. Thus $(gf)^*$ is homotopic to $1\colon P_\bullet \longrightarrow P_\bullet$, and similarly we get that $(gf)^*$ is homotopic to the identity on $Q_\bullet$.
It follows from (4.2.3) that on the homology level g^* is the inverse of f^*, proving our claim. □

5.3.6 Corollary. *Let $\{P_n, d_n\}$ be a projective resolution of an A-module M, and let $K_i = \ker(d_i) \subseteq P_i$ for all $i \geq 0$. Then we have*

$$\mathrm{Ext}^{n+1}_A(M, N) = \mathrm{Ext}^{n-i}(K_i, N),$$

for any A-module N, and any $i \geq 0$.

Proof. Fix an integer $i \geq 0$. Let $Q_n = P_{n+i+1}$, and $\epsilon_n = d_{n+i+1}$. After this renaming we have the projective resolution of K_i;

$$Q_n \xrightarrow{\epsilon_n} Q_{n-1} \xrightarrow{\epsilon_{n-1}} \cdots \longrightarrow Q_0 \xrightarrow{\epsilon_0} K_i \longrightarrow 0.$$

We apply $\mathrm{Hom}_A(-, N)$ to the resolution of K_i we obtain from this sequence. Then by taking homology we obtain, by definition,

$$\mathrm{Ext}^n_A(K_i, N) = \ker(\epsilon^*_{n+1})/\mathrm{im}(\epsilon^*_n),$$

for all $n \geq 0$. We have $\epsilon_{n-i+1} = d_{n+2}$, hence the homology module above in degree $n - i$ is $\mathrm{Ext}^{n+1}_A(M, N)$. $\square$

5.3.7 Corollary. *If M is a projective A-module then* $\mathrm{Ext}^n_A(M, N) = 0$ *for any A-module N $(n > 0)$.*

Proof. If M is projective we can take the projective resolution $P_0 = M$, which have only trivial homology modules for all $n \geq 1$. Thus $\mathrm{Ext}^n_A(M, N) = 0$ for all $n \geq 1$. $\square$

5.3.8 Lemma. *Let* $0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$ *be a short exact sequence of A-modules. Then there exist projective resolutions $P_\bullet(M_i)$ of M_i, for $i = 1, 2, 3$, and chain maps giving a short exact sequence of complexes*

$$0 \longrightarrow P_\bullet(M_1) \longrightarrow P_\bullet(M_2) \longrightarrow P_\bullet(M_3) \longrightarrow 0.$$

Proof. Let $d^R_0 \colon R_0 \longrightarrow M_3$ be a surjective homomorphism from a projective module R_0. Since $M_2 \longrightarrow M_3$ is assumed surjective and R_0 is projective there exists a lifting $\tilde{d}_0 \colon R_0 \longrightarrow M_2$ of d^R_0. Let $d^P_0 \colon P_0 \longrightarrow M_1$ be a surjective homomorphism from a projective module P_0 to M_1, and consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & P_0 \arrow[r] \arrow[d, "d^P_0"] & P_0 \oplus R_0 \arrow[r] \arrow[d, "d^Q_0"] & R_0 \arrow[r] \arrow[d, "d^R_0"] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "f"] & M_2 \arrow[r, "g"] & M_3 \arrow[r] & 0,
\end{tikzcd}$$

where $d^Q_0(x, y) := f(d^P_0(x)) + \tilde{d}_0(y)$. The diagram is commutative, and the upper horizontal sequence is a short exact sequence of projective modules P_0, Q_0 and R_0, where $Q_0 = P_0 \oplus R_0$. The kernels of the three vertical maps we denote by K_1, K_2 and K_3. It is readily checked that the induced sequence

$$0 \longrightarrow K_1 \longrightarrow K_2 \longrightarrow K_3 \longrightarrow 0$$

is exact. Applying the above argument to this short exact sequence gives projective modules P_1, Q_1 and R_1. The inductive process described gives projective modules $P_\bullet, Q_\bullet$ and $R_\bullet$ and chain maps making a short exact sequence $0 \longrightarrow P_\bullet \longrightarrow Q_\bullet \longrightarrow R_\bullet \longrightarrow 0$. Furthermore, each of the three complexes are exact by construction, hence giving projective resolutions of M_1, M_2 and M_3. $\square$

Split sequences

A short exact sequence

$$0 \longrightarrow M_1 \xrightarrow{f} N \xrightarrow{g} M_2 \longrightarrow 0$$

is called *split* if N is isomorphic to $M_1 \oplus M_2$, and under this isomorphism the morphism $f\colon M_1 \longrightarrow M_1 \oplus M_2$ is the canonical inclusion, and $g\colon M_1 \oplus M_2 \longrightarrow M_2$ is the canonical projection.

5.3.9 Lemma. *Let* $0 \longrightarrow M_1 \xrightarrow{f} N \xrightarrow{g} M_2 \longrightarrow 0$ *be a short exact sequence of A-modules. The sequence is split if and only if g has a section; that is there exists an A-module homomorphism $s\colon M_2 \longrightarrow N$ such that $g \circ s = \mathrm{id}_{M_2}$. In particular this always holds if M_2 is a projective module.*

Proof. If the sequence splits then clearly g has a section. Conversely, if g has a section then N becomes naturally isomorphic to the direct sum $M_1 \oplus M_2$ in the following way. Define the A-module homomorphism $\varphi\colon M_1 \oplus M_2 \longrightarrow N$ by sending $(x, y) \mapsto f(x) + s(y)$, where $s\colon M_2 \longrightarrow N$ is one fixed section of g. To see that this map is injective, we assume (x, y) is in the kernel. Then $f(x) + s(y) = 0$, hence $g(f(x) + s(y)) = 0 + y$, so $y = 0$. But as f is injective it also follows that $x = 0$. Hence φ is injective. To establish surjectivity of φ, let $z \in N$ be an element. Let $y = g(z) \in M_2$. Then we have that $g(z - s(y)) = g(z) - g(s(y)) = y - y = 0$. Hence the element $z - s(y) = f(x) \in N$, for some $x \in M_1$. We then get that (x, y) is mapped to $\varphi(x, y) = f(x) + s(y) = z$, and φ is surjective.
Finally if M_2 were projective then the identity morphism $\mathrm{id}\colon M_2 \longrightarrow M_2$ combined with the defining property of M_2 being projective, would give a lifting to $s\colon M_2 \longrightarrow N$. Such a lifting would be a section. $\square$

5.3.10 Proposition. *Let* $0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$ *be a short exact sequence of A-modules. For each A-module N we get an induced long exact sequence of Ext-modules*

$$0 \longrightarrow \mathrm{Hom}_A(M_3, N) \longrightarrow \mathrm{Hom}_A(M_2, N) \longrightarrow \mathrm{Hom}_A(M_1, N) \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathrm{Ext}^n_A(M_3, N) \longrightarrow \mathrm{Ext}^n_A(M_2, N) \longrightarrow \mathrm{Ext}^n_A(M_1, N) \longrightarrow \cdots .$$

Proof. By Lemma (5.3.8) we can find projective resolutions of the modules M_i $(i = 1, 2, 3)$, that fits into a short exact sequence

$$0 \longrightarrow P_\bullet \longrightarrow Q_\bullet \longrightarrow R_\bullet \longrightarrow 0.$$

We then apply $\mathrm{Hom}_A(-, N)$ to all the modules appearing in the three resolutions above. For each degree n we get the horizontal sequence

$$0 \longrightarrow \mathrm{Hom}_A(R_n, N) \longrightarrow \mathrm{Hom}_A(Q_n, N) \longrightarrow \mathrm{Hom}_A(P_n, N) \longrightarrow 0. \tag{5.3}$$

By Lemma (5.3.9) we have that $Q_n = P_n \oplus R_n$, and in particular that

$$\mathrm{Hom}_A(Q_n, N) = \mathrm{Hom}_A(P_n, N) \oplus \mathrm{Hom}_A(R_n, N).$$

It follows that the sequence (5.3) is exact. Thus after applying $\mathrm{Hom}_A(-, N)$ we obtain three complexes $P_\bullet^\wedge, Q_\bullet^\wedge$ and $R_\bullet^\wedge$ that form a short exact sequence

$$0 \longrightarrow R_\bullet^\wedge \longrightarrow Q_\bullet^\wedge \longrightarrow P_\bullet^\wedge \longrightarrow 0.$$

By Theorem (4.1.3) we get a long exact sequence of homology modules, and by (5.3.5) these homology modules are precisely the Ext-modules.

We have then proved that this particular resolution give a long exact sequence of homology modules. By Proposition (5.3.5) any other resolution would give the same homology modules, but we also need to check that the associated module homomorphism are independent of the resolution. However, by the Comparison Theorem (5.3.2) it follows that the actual maps between possible different resolutions are homotopic, hence giving the same maps on homology. □

5.3.11 Example. Let A be a field. Any module M over a field is a vector space, so in particular M is free, and projective. Consequently $\mathrm{Ext}^n_A(M, N) = 0$ with $n > 0$, for any vector space N.

5.3.12 Example. Let M be a finitely generated $\mathbf{Z}$-module. We have that $\mathrm{Ext}^n_A(M, N) = 0$ for $n \geq 2$. Indeed, we have by Proposition (5.2.3) that a finitely generated $\mathbf{Z}$-module has a free resolution of length at most one.

5.3.13 Example. Let $\mathbf{Z}_n = \mathbf{Z}/(n)$ denote the cyclic group of n-elements. We then have the projective resolution

$$0 \longrightarrow \mathbf{Z} \xrightarrow{n} \mathbf{Z} \longrightarrow \mathbf{Z}_n \longrightarrow 0.$$

For any $\mathbf{Z}$-module N we get the induced long exact sequence

$$\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}, N) \xrightarrow{n^*} \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}, N) \longrightarrow \mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}_n, N) \longrightarrow \mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}, N).$$

Since $\mathbf{Z}$ is free, and in particular projective, we have that the rightmost module is zero. We have furthermore that $\mathrm{Hom}_A(A, R) = R$, hence the two left most groups are identified with N. The map

$$n^* \colon N \longrightarrow N$$

is multiplication with n; that is $x \mapsto nx$. The cokernel of the map n^* is therefore the Ext-module, $N/nN = \mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}_n, N)$. In particular this applies to our standard Example (3.1.2).

5.4 Injective resolutions

One of the reasons to introduce the projective modules, and not only stick to free modules, is that there is a dual counter part. Let N be an A-module, and assume that we have an exact sequence

$$0 \longrightarrow N \longrightarrow I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \cdots,$$

with injective modules I_n. Then $\{I_n, d_n\}$ is an *injective* resolution of N. Let M be another A-module, and apply $\mathrm{Hom}_A(M, -)$ to one injective resolution of N. We then have the complex

$$0 \longrightarrow \mathrm{Hom}_A(M, I_0) \xrightarrow{d_{0,*}} \mathrm{Hom}_A(M, I_1) \xrightarrow{d_{1,*}} \cdots$$

As in the situation with projective modules a similar argument (cf. Exercise (5.5.15)) shows that the homology modules are independent of the resolution. We define the homology modules

$$\mathrm{Ext}^A_n(M, N) := \ker(d_{n,*})/\mathrm{im}(d_{n-1,*}).$$

The dual statements for results of the previous section follows mutatis mutandi. We encourage the reader to at least state all the previous statements for injective modules and injective resolutions (see Exercises 5.5.14 -5.5.18).

5.4.1 Theorem. *For any A-modules M and N, and any integer $n \geq 0$ we have* $\mathrm{Ext}^A_n(M, N) = \mathrm{Ext}^n_A(M, N)$.

Proof. Let $\{P_n, d_n\}$ be a projective resolution of M, and let $K_i = \ker(d_i)$. We then have the short exact sequence

$$0 \longrightarrow K_{i+1} \longrightarrow P_{i+1} \longrightarrow K_i \longrightarrow 0,$$

for all $i \geq 0$. In fact we let $K_{-1} = M$, and consider the above sequence for all $i \geq -1$. Similarly, we let $\{Q_n, e_n\}$ be an injective resolution of N, and we let C_{j+1} be the kernel of $e_j \colon Q_j \longrightarrow Q_{j+1}$. We then get the short exact sequence

$$0 \longrightarrow C_j \longrightarrow Q_j \longrightarrow C_{j+1} \longrightarrow 0,$$

for all $j \geq 0$, where $N = C_0$. We fix two integers $i \geq -1, j \geq 0$ and form the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_A(K_i, C_j) \arrow[r] \arrow[d] & \mathrm{Hom}_A(K_i, Q_j) \arrow[r] \arrow[d] & \mathrm{Hom}_A(K_i, C_{j+1}) \arrow[r] \arrow[d] & F \\
0 \arrow[r] & \mathrm{Hom}_A(P_{i+1}, C_j) \arrow[r] \arrow[d] & \mathrm{Hom}_A(P_{i+1}, Q_j) \arrow[r] \arrow[d] & \mathrm{Hom}_A(P_{i+1}, C_{j+1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(K_{i+1}, C_j) \arrow[r] \arrow[d] & \mathrm{Hom}_A(K_{i+1}, Q_j) \arrow[r] \arrow[d] & \mathrm{Hom}_A(K_{i+1}, C_{j+1}) \arrow[r] \arrow[d] & F' \\
 & E & 0 & E' & .
\end{tikzcd}$$

The A-modules E, E' and F, F' are cokernels, making the related sequences exact. Since P_{i+1} is projective, we get by Corollary (5.3.7) that the middle horizontal row is exact. And similarly, by Exercise (5.5.18), the middle vertical column is exact as Q_j is injective. Thus the whole diagram above is consist of exact rows and columns.

Applying the Snake Lemma (2.2.3) to the diagram consisting of rows two and three, we get that $E = F$. Since P_{i+1} is projective we that $\mathrm{Ext}^1_A(P_{i+1}, C_j) = 0$. Consequently, by the long exact sequence (5.3.10) we get that $E = \mathrm{Ext}^1_A(K_i, C_j)$. Similarly, we have that $F = \mathrm{Ext}^A_1(K_i, C_j)$, thus

$$\mathrm{Ext}^1_A(K_i, C_j) = \mathrm{Ext}^A_1(K_i, C_j) \tag{5.4}$$

for all $i \geq -1$ and $j \geq 0$. A diagram chase (see Exercise (5.5.8)) shows that $E' = F'$, hence

$$\mathrm{Ext}^1_A(K_i, C_{j+1}) = \mathrm{Ext}^A_1(K_{i+1}, C_j), \tag{5.5}$$

for all i and j. To prove the theorem we note that by Lemma (5.3.6) we have

$$\mathrm{Ext}^{n+1}_A(M, N) = \mathrm{Ext}^1_A(K_{n-1}, N).$$

Substitute $N = C_0$. By using (5.5) and then (5.4) repeatedly we can decrease the index of the first factor K_{n-1}, and increase the index of the second factor C_0. Using this shifting process n times we arrive at

$$\mathrm{Ext}^1_A(K_{n-1}, C_0) = \mathrm{Ext}^1_A(K_{-1}, C_n).$$

Then by using the identification of (5.4), and finally using the injective version of Lemma (5.3.6) (see Exercise(5.5.17)) we get that

$$\mathrm{Ext}^1_A(K_{-1}, C_n) = \mathrm{Ext}^A_1(K_{-1}, C_n) = \mathrm{Ext}^{n+1}_A(K_{-1}, N).$$

As $K_{-1} = M$ by definition, we have proven the theorem. □

5.4.2 Corollary. *The A-module M is projective if and only if we have that $\mathrm{Ext}^1_A(M, N) = 0$ for all A-modules N.*

Proof. If M is projective we have by Corollary (5.3.7) that $\mathrm{Ext}^n_A(M, N) = 0$ for all $n \geq 1$, and in particular for $n = 1$.

To prove the converse, let $0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$ be any short sequence of A-modules. By taking injective resolutions we get by a result similar to Proposition (5.3.10) an induced long exact sequence of the associated homology modules $\mathrm{Ext}^*_A(M, -)$. By the Theorem we have that $\mathrm{Ext}^A_n(M, N) = \mathrm{Ext}^n_A(M, N)$ for all modules M and N. By assumption the higher Ext-modules are zero, which means that the long exact sequence consists of three terms only, the degree zero homology modules. We have that $\mathrm{Ext}^0_A(M, N) = \mathrm{Hom}_A(M, N)$. Consequently the assumption that the modules $\mathrm{Ext}^1_A(M, N) = 0$ for all modules N, implies that the functor $\mathrm{Hom}_A(M, -)$ is exact. By Theorem (3.2.3) we get that M is projective. □

5.5 Exercises

5.5.1. Compute a free finite resolution of $\mathbf{Z}/(2) \oplus \mathbf{Z}/(30) \oplus \mathbf{Z}$.

5.5.2. Show that we have an exact sequence

$$0 \longrightarrow \mathbf{Z}/(p) \longrightarrow \mathbf{Z}/(p^2) \longrightarrow \mathbf{Z}/(p) \longrightarrow 0,$$

which does not split.

5.5.3. Let M be an A-module, and let $F_\bullet$ and $G_\bullet$ be two finite free resolutions of M. Let n be the length of the resolution $F_\bullet$ and let m be the length of the resolution $G_\bullet$. Show that there is an equality

$$\sum_{i=0}^{n} (-1)^i \mathrm{rank} F_i = \sum_{j=0}^{m} (-1)^j \mathrm{rank} G_j.$$

As the quantity described above is independent of the finite free resolution of M we denote this by $\chi(M)$, and this number is called the *Euler-Poincare characteristic* of M.

5.5.4. Show that the sequence

$$0 \longrightarrow m\mathbf{Z}/(mn) \longrightarrow \mathbf{Z}/(mn) \longrightarrow n\mathbf{Z}/(mn) \longrightarrow 0$$

splits if and only if $(m, n) = 1$.

5.5.5. Show that $\mathrm{Ext}^0_A(M, N) = \mathrm{Hom}_A(M, N)$.

5.5.6. Show that the elements of $\mathrm{Ext}^1_A(M, N)$ corresponds to extensions E of the form

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0,$$

that is exact sequences with N and M as the ends.

5.5.7. Let $j\colon M \longrightarrow N$ be an injective homomorphism. Show that $j(M)$ is a direct summand of N if and only if j has a section $s\colon N \longrightarrow M$. Show, furthermore, that if so, then $N = j(M) \oplus \ker(s)$.

5.5.8. Given a commutative diagram of A-modules

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"] & N_1 \arrow[d] \\ N_2 \arrow[r] & Q \end{tikzcd}$$

Assume that the f and g are surjective homomorphisms. Show that cokernel $N_1 \longrightarrow Q$ is naturally isomorphic to cokernel $N_2 \longrightarrow Q$. Note that two quotient modules of a fixed module Q are equal if and only if the kernels of the quotient maps coincide.

5.5.9. Determine the group $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/(8), \mathbf{Z}/(12))$.

5.5.10. Let p be a prime number. Show that there are exactly p different extensions E of the form

$$0 \longrightarrow \mathbf{Z}/(p) \longrightarrow E \longrightarrow \mathbf{Z}/(p) \longrightarrow 0.$$

Show that there are only two different choices for E, namely $\mathbf{Z}/(p) \oplus \mathbf{Z}/(p)$ and $\mathbf{Z}/(p^2)$. Why does this not contradict Excercise (5.5.6)?

5.5.11. We have that $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/(4), \mathbf{Z}/(4)) = G$ is an $\mathbf{Z}$-module, and in particular a group. By Excercise (5.5.6) we have that the short exact sequence

$$0 \longrightarrow \mathbf{Z}/(4) \longrightarrow \mathbf{Z}/(16) \longrightarrow \mathbf{Z}/(4) \longrightarrow 0$$

corresponds to an element in the group G. Construct the extension that corresponds to its inverse.

5.5.12. Compute $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/(p), \mathbf{Z})$.

5.5.13. Show that if M is torsion free, then $\mathrm{Ext}^1_{\mathbf{Z}}(M, \mathbf{Z})$ is divisible. And, if M is divisible, then $\mathrm{Ext}^1_{\mathbf{Z}}(M, \mathbf{Z})$ is torsion free.

5.5.14. Let $I_\bullet$ and $J_\bullet$ be two injective resolution of a module N. Show that there exists a chain map $f\colon I_\bullet \longrightarrow J_\bullet$, and that any other chain map is homotopic to f.

5.5.15. Use the previous exercise to show that the homology groups $\mathrm{Ext}^A_n(M, N)$ are independent of the injective resolution of N.

5.5.16. Use the previous exercise to show that if we have a short exact sequence

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

of A-modules, then we get for any A-module M an long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^A_{n+1}(M, N_3) \longrightarrow \mathrm{Ext}^A_n(M, N_1) \longrightarrow \mathrm{Ext}^A_n(M, N_2) \longrightarrow \cdots$$

5.5.17. Let $\{I_n, d_n\}$ be an injective resolution of N, and let $C_i = \ker(d_i)$ for all $i \geq 0$. Use the definition directly to show that, for any module M, we have

$$\mathrm{Ext}^A_{n+i}(M, N) = \mathrm{Ext}^A_n(M, C_i).$$

5.5.18. Show that an A-module N is injective if and only if we have that $\mathrm{Ext}^A_1(M, N) = 0$ for all A-modules M.

Chapter 6

Tensor product and Tor

6.1 The construction of the tensor product

The tensor product of two modules is an important object, but not an easy one. We start by giving its construction.

Let M and N be two A-modules. Let $T(M,N)$ denote the free module indexed by the elements in the product $M \times N$, that is

$$T(M,N) = \bigoplus_{(x,y)\in M\times N} Ae_{x,y},$$

where $e_{x,y}$ is simply notation tagging the component that corresponds to the element $(x,y) \in M \times N$. This is an enormous module!

Consider the following family of relations

$$\begin{aligned} e_{x+x',y} - e_{x,y} - e_{x',y}, \\ e_{x,y+y'} - e_{x,y} - e_{x,y'}, \\ e_{ax,y} - e_{x,ay}, \\ a \cdot e_{x,y} - e_{ax,y}, \end{aligned}$$

for all $a \in A$, all $x, x' \in M$ and all $y, y' \in N$. These relations generate an A-submodule $R \subseteq T(M,N)$, and the tensor product is defined as the quotient module. That is,

$$M \otimes_A N := T(M,N)/R,$$

and referred to as the tensor product of M and N, over A. The class of a basis element $e_{x,y}$ in $M \otimes_A N$ is denoted $x \otimes y$, and such an element is referred to as a *pure* tensor. A general element $z \in M \otimes_A N$ is called a *tensor*, and is a finite linear combinations of the form

$$z = \sum_{i=1}^{n} x_i \otimes y_i,$$

with $x_i \in M$ and $y_i \in N$, for $i = 1, \dots, n$.

6.1.1 Example. Consider the $\mathbf{Z}$-modules $M = \mathbf{Z}/m\mathbf{Z}$ and $N = \mathbf{Z}/n\mathbf{Z}$. Assume that the two numbers m and n are coprime, which means that the biggest common divisor of the two numbers are 1. In other words there exist numbers a and b such that $1 = am + bn$. Consider now the tensor product

$$\mathbf{Z}/m\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/n\mathbf{Z},$$

which we claim is the zero module. To see this it suffices to show that the element $1 \otimes 1 = 0$. We have that $1 \otimes 1$ is the class of the element

$$e_{1,1} = e_{am+bn,1} = e_{am,1} + e_{bn,1}.$$

We have that $m = 0$ in $\mathbf{Z}/m\mathbf{Z}$, and as $0 \otimes x = 0$ in any tensor product $M \otimes_A N$, we get that

$$e_{1,1} = e_{0,1} + e_{b,n} = e_{0,1} + e_{b,0}.$$

Thus $1 \otimes 1 = 0 \otimes 1 + b \otimes 0 = 0$. And we have that the tensor product $\mathbf{Z}/m\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/n\mathbf{Z} = 0$.

6.1.2 Proposition. *Let $\{M_i\}_{i \in I}$ be a collection of A-modules. For any A-module N we have that*

$$\big(\oplus_{i \in I} M_i\big) \otimes_A N = \oplus_{i \in I}(M_i \otimes_A N).$$

Proof. Using the notation form the previous section. It is clear that

$$T(\oplus_{i \in I} M_i, N) = \oplus_{i \in I} T(M_i, N).$$

Let R be the submodule defining $(\oplus_{i \in I} M_i) \otimes_A N$as a quotient of $T(\oplus_{i \in I} M_i, N)$. And let $R' = \oplus_{i \in I} R_i$, where $R_i \subseteq T(M_i, N)$ is the submodule with quotient $M_i \otimes_A N$. Since any element in a direct sum is a finite sum, it is clear that $R = R'$, and the proposition follows.

□

6.2 Universal property of the tensor product

An A-module homomorphism $f\colon M \times N \longrightarrow P$ is *bilinear* if it is linear in both factors. That is, for fixed $x \in M$ the induced map $f(x, -)\colon N \longrightarrow P$ is A-linear, for all $x \in M$. And, similarly for all $y \in N$ the induced map $f(-, y)\colon M \longrightarrow P$ is A-linear.

6.2.1 Theorem (Universal property of tensor product)**.** *Let M and N be two A-modules. The natural map $F\colon M \times N \longrightarrow M \otimes_A N$ sending $(x, y) \mapsto x \otimes y$ is bilinear. It is furthermore the universal bilinear map from $M \times N$; Let $f\colon M \times N \longrightarrow P$ be any bilinear map to an A-module P. Then there exists a unique A-module homomorphism $\varphi\colon M \otimes_A N \longrightarrow P$ such that $f = \varphi F$.*

Proof. The map of sets $M \times N \longrightarrow T(M, N)$ sending $(x, y) \mapsto e_{x,y}$ is well-defined. Composed with the projection $T(M, N) \longrightarrow M \otimes_A N$ gives a well-defined map $M \times N \longrightarrow M \otimes_A N$. A routine verification shows that the map indeed is bilinear. Let $f\colon M \times N \longrightarrow P$ be a bilinear map. Define the A-module homomorphism $\Phi\colon T(M, N) \longrightarrow P$ by sending $e_{x,y} \mapsto f(x, y)$. Since the map f is assumed to be bilinear the A-module homomorphism Φ will have the relations $R \subseteq T(M, N)$ in its kernel, and consequently we get a induced homomorphism $\varphi\colon M \otimes_A N \longrightarrow P$ factorizing $f\colon M \times N \longrightarrow P$. Let $\varphi'\colon M \otimes_A N \longrightarrow P$ be another A-module homomorphism factorizing $f\colon M \times N \longrightarrow P$. We then would have, for any $(x, y) \in M \times N$ that

$$f(x, y) = \varphi(x \otimes y) = \varphi'(x \otimes y).$$

Since the pure tensors generate the tensor product, it follows that $\varphi = \varphi'$. That is, the factorization is unique.

□

6.2.2 Proposition. *The natural map $M \otimes_A A \longrightarrow M$ sending $(x, a) \mapsto ax$, is an isomorphism.*

Proof. We have the bilinear map $M \times A \longrightarrow M$ sending $(x, a) \mapsto ax$. Let $c\colon M \otimes_A A \longrightarrow M$ denote the induced A-module homomorphism. Let $f\colon M \times A \longrightarrow N$ be any bilinear map to an arbitrary A-module N. Then in particular the map

$$\varphi = f(-, 1)\colon M \longrightarrow N$$

is an A-module homomorphism. Let $F\colon M \otimes_A A \longrightarrow N$ denote the A-module homomorphism corresponding to the bilinear map f. As we have

$$x \otimes a = ax \otimes 1,$$

it follows that $F = \varphi c$. Thus any A-module homomorphism from $M \otimes_A A$ has a factorization through $c \colon M \otimes_A A \longrightarrow M$. In particular, setting $N = M \otimes_A A$ and letting $F = \mathrm{id}$ be the identity map, we get the existence of an A-module homomorphism φ being the inverse of c.

$\square$

6.2.3 Corollary. *Let M be a free module with basis $x_1, \ldots, x_m$, and let N be a free module with basis $y_1, \ldots, y_n$. Then the pure tensors $x_i \otimes y_j$ with $1 \leq i \leq m, 1 \leq j \leq n$ form an A-module basis of $M \otimes_A N$. In particular $M \otimes_A N$ is free.*

Proof. We have $M = \oplus_{i=1}^m Ax_i$. By Proposition (6.1.2) we have

$$M \otimes_A N = \oplus_{i=1}^m Ax_i \otimes N.$$

As we have the obvious isomorphism $P \otimes_A Q = Q \otimes_A P$, for any A-modules P and Q, we can also apply Proposition (6.1.2) on the other factor as well. That gives

$$M \otimes_A N = M \otimes_A (\oplus_{j=1}^m Ay_j) = \oplus_{i,j} Ax_i \otimes_A Ay_j.$$

By the Proposition above, we get that $Ax_i \otimes_A Ay_j = A(x_i \otimes y_j) = A$.

$\square$

6.3 Right exactness of tensor product

6.3.1 Lemma. *Let $f \colon M \longrightarrow M'$ and $g \colon N \longrightarrow N'$ be two A-module homomorphism. Then we have an induced A-module homomorphism*

$$f \otimes g \colon M \otimes_A N \longrightarrow M' \otimes_A N',$$

sending $x \otimes y \mapsto f(x) \otimes g(y)$.

Proof. The map $M \times N \longrightarrow M' \otimes_A N'$ sending $(x, y) \mapsto f(x) \otimes g(y)$ is well-defined, and bilinear. The induced map of tensor products then follows from the universal properties of the tensor product.

$\square$

6.3.2 Proposition. *Let the following be a short exact sequence of A-modules*

$$0 \longrightarrow N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3 \longrightarrow 0$$

Then, for any A-module M, the following induced sequence is also exact

$$M \otimes_A N_1 \xrightarrow{1\otimes f} M \otimes_A N_2 \xrightarrow{1\otimes g} M \otimes_A N_3 \longrightarrow 0.$$

In other words, $(M \otimes_A -)$ is a right exact covariant functor.

Proof. By Lemma (6.3.1) there exist A-module homomorphisms as described in the Proposition, from where it follows that we have a complex. Surjectivity of the rightmost map is left for the reader to check. The tricky part is to show that

$$\mathrm{Ker}(1 \otimes g) \subseteq \mathrm{Im}(1 \otimes f).$$

Let $I \subseteq M \otimes_A N_2$ denote the image of the A-module homomorphism $1 \otimes f$. As we have that the sequence is a complex we have that $1 \otimes g$ has a factorization through

$$\overline{1 \otimes g} \colon M \otimes_A N_2/I \longrightarrow M \otimes_A N_3.$$

We need to show that $\overline{1 \otimes g}$ is injective. This we will do by showing that the homomorphism $\overline{1 \otimes g}$ has a section. For any $y' \in N_3$, let $y \in N_2$ be an element such that $g(y) = y'$. Define the map

$$M \times N_3 \longrightarrow M \otimes N_2/I,$$

by sending $(x, y') \mapsto \overline{x \otimes y}$, where $\overline{x \otimes y}$ denotes the class of the tensor $x \otimes y$ in $M \otimes_A N_3/I$. The reader is encouraged to check that the map is independent of the choice of pre-image y of y', making it a well-defined map.

The map is clearly bilinear, hence we obtain by the universal properties of the tensor product an A-module homomorphism

$$s \colon M \otimes_A N_3 \longrightarrow M \otimes_A N_2/I.$$

By construction we have that, for any $y' \in N_3$ any $x \in M$ that

$$\overline{1 \otimes g}(s(x \otimes y')) = \overline{1 \otimes g}(\overline{x \otimes y}) = x \otimes y'.$$

Since the *pure* tensors $x \otimes y'$ generate the A-module $M \otimes_3 N$ we have proven that s is a section of $\overline{1 \otimes g}$. Hence $\overline{1 \otimes g}$ is injective, that is $\mathrm{Ker}(1 \otimes g) \subseteq \mathrm{Im}(1 \otimes f)$. $\square$

6.3.3 Example. The tensor product is not left exact in general, as shown by the following example. Consider the short exact sequence of $\mathbf{Z}$-modules

$$0 \longrightarrow \mathbf{Z} \xrightarrow{2} \mathbf{Z} \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow 0.$$

Tensor this sequence with $(- \otimes_{\mathbf{Z}} \mathbf{Z}/2\mathbf{Z})$. By Proposition (6.2.2) we have the identity $M \otimes_A A = M$, hence we get the sequence

$$\mathbf{Z}/2\mathbf{Z} \xrightarrow{2} \mathbf{Z}/2\mathbf{Z} \longrightarrow \mathbf{Z}/2\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/2\mathbf{Z} \longrightarrow 0.$$

The multiplication by 2 map equals the zero map. So in particular the sequence is not left exact, and we also obtain that

$$\mathbf{Z}/2\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/2\mathbf{Z} = \mathbf{Z}/2\mathbf{Z}.$$

6.3.4 Proposition. *Let m and n be two integers, and let $c = \mathrm{GCD}(m, n)$ denote their greatest common divisor. Then we have*

$$\mathbf{Z}/(m) \otimes_{\mathbf{Z}} \mathbf{Z}/(n) = \mathbf{Z}/(c).$$

Proof. Using the right exactness of tensor product, and the identification of Proposition (6.2.2), we obtain the commutative diagram of surjective $\mathbf{Z}$ module homomorphisms

$$\begin{tikzcd}
\mathbf{Z} \arrow[r] \arrow[d] & \mathbf{Z}/(m) \arrow[d] \\
\mathbf{Z}/(n) \arrow[r] & \mathbf{Z}/(n) \otimes_{\mathbf{Z}} \mathbf{Z}/(m).
\end{tikzcd}$$

In particular we have $\mathbf{Z}/(m) \otimes_{\mathbf{Z}} \mathbf{Z}/(n)$ equals to $\mathbf{Z}/I$, for some ideal $I \subseteq \mathbf{Z}$. From the commutative diagram above we have $(m) \subseteq I$, and $(n) \subseteq I$. Furthermore, the *smallest* ideal containing both (m) and (n) is the ideal generated by (c), where c is their greatest common divisor. We must therefore have $(c) \subseteq I$, and in particular we get an induced surjective $\mathbf{Z}$-module homomorphism

$$\mathbf{Z}/(c) \longrightarrow \mathbf{Z}/(m) \otimes_{\mathbf{Z}} \mathbf{Z}/(n). \tag{6.1}$$

We have furthermore, a natural bilinear map $\mathbf{Z}/(m) \times_{\mathbf{Z}} \mathbf{Z}/(n) \longrightarrow \mathbf{Z}/(c)$ taking $(x, y) \mapsto x \cdot y$. Note that this map is well-defined because the ideal (c) contains both (m) and (n). It follows that we get an induced $\mathbf{Z}$-linear homomorphism $\mathbf{Z}/(m) \otimes_{\mathbf{Z}} \mathbf{Z}/(n) \longrightarrow \mathbf{Z}/(c)$. This homomorphism is the inverse of (6.1).

$\square$

6.4 Flat modules

As we have seen, tensoring does not preserve short exactness. Those modules M, however, that do preserve exactness are of particular interest.

6.4.1 Definition. *An A-module M such that for any short exact sequence of A-modules $0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$ the induced sequence*

$$0 \longrightarrow N_1 \otimes_A M \longrightarrow N_2 \otimes_A M \longrightarrow N_3 \otimes_A M \longrightarrow 0$$

is exact, *is called a* flat-*module.*

6.4.2 Example. The ring A considered as an A-module is flat. This is indeed the case as $M \otimes_A A = M$ by Proposition (6.2.2).

6.4.3 Example. Let F be a free module, that is $F = \oplus_{i \in I} A$. For any module M we have by Proposition (6.1.2) and Proposition (6.2.2) that $F \otimes_A M = \oplus_{i \in I} M$. It follows that free modules are flat. In particular we have that vector spaces are always flat.

6.4.4 Example. Consider the $\mathbf{Z}$-module $M = \mathbf{Z}/(n)$, for some integer $n > 1$. By replacing the 2 in Example (6.1.1) we get that M is not flat. Note that when $n = 1$ the module is the zero module, which is flat by trivial reasons. And when $n = 0$ the module $M = \mathbf{Z}$ is free, hence flat.

6.4.5 Proposition. *Let M be an A-module. The following are equivalent*

(1) The module M is flat.

(2) The induced map $N_1 \otimes_A M \longrightarrow N_2 \otimes_A M$ is injective, for any injective map $N_1 \longrightarrow N_2$ of A-modules.

(3) For any injective map $N_1 \longrightarrow F$ with F a free A-module, the induced map $N_1 \otimes_A M \longrightarrow F \otimes_A M$ is injective.

Proof. It it is clear that (1) is equivalent with (2), and that (2) implies (3). We will show that (3) implies (2). Let $0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$ be a short exact sequence of A-modules. We can find free modules F_2 and F_3 making a commutative diagram

$$\begin{array}{ccc} F_2 & \xrightarrow{f} & F_3 \\ \big\downarrow{\scriptstyle f_2} & & \big\downarrow{\scriptstyle f_3} \\ N_2 & \longrightarrow & N_3, \end{array}$$

with surjective homomorphisms. Let $C = \mathrm{Ker}(f)$ and let $K_i \subseteq F_i$ denote the kernel of f_i, $(i = 2, 3)$. There is an induced surjective map $C \longrightarrow N_1$, and we let Z denote the kernel of that homomorphism. From the right exactness of the tensor product we obtain the following commutative diagram and exact rows

$$\begin{array}{ccccccc} Z \otimes_A M & \longrightarrow & K_2 \otimes_A M & \longrightarrow & K_3 \otimes_A M & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ C \otimes_A M & \longrightarrow & F_2 \otimes_A M & \longrightarrow & F_3 \otimes_A M & \longrightarrow & 0. \end{array}$$

By assumption M satisfies the condition (3), so in particular the bottom horizontal sequence in (6.4) is injective to the left. The Snake Lemma then gives that we have a short exact sequence of the co-kernels of the three vertical maps in the diagram. By the right exactness of the tensor product we obtain that the co-kernel of the vertical maps is the sequence

$$N_1 \otimes_A M \longrightarrow N_2 \otimes_A M \longrightarrow N_3 \otimes_A M.$$

Injectivity of the map $N_1 \otimes_A M \longrightarrow N_2 \otimes_A M$ follows as the right most vertical sequence in (6.4) is injective by the assumption of M. Thus, we have proven that (3) implies (2) (and in fact that (3) implies (1)).

□

6.4.6 Proposition. *Let $M_1, \ldots, M_n$ be a finite collection of A-modules, and let $M = \oplus_{i=1}^n M_i$ denote their direct sum. Then M is flat if and only each summand is flat.*

Proof. We can by induction reduce to the situation to having two summands only, $M = M_1 \oplus M_2$. For any A-module Q the sequence

$$0 \longrightarrow M_1 \otimes_A Q \longrightarrow (M_1 \oplus M_2) \otimes_A Q \longrightarrow M_2 \otimes_A Q \longrightarrow 0$$

is exact since the middle term is $M_1 \otimes_A Q \oplus M_2 \otimes_A Q$ (Proposition (6.1.2)). Let $N \longrightarrow F$ be a injective A-module homomorphism, with F free, and let Q denote its cokernel. Consider the commutative diagram

$$
\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& M_1 \otimes_A N \arrow[r] \arrow[d] & M_1 \otimes_A F \arrow[r] \arrow[d] & M_1 \otimes_A Q \arrow[r] \arrow[d] & 0 \\
& (M_1 \oplus M_2) \otimes N \arrow[r] \arrow[d] & (M_1 \oplus M_2) \otimes F \arrow[r] \arrow[d] & (M_1 \oplus M_2) \otimes Q \arrow[r] \arrow[d] & 0 \\
& M_2 \otimes_A N \arrow[r] \arrow[d] & M \otimes_A F \arrow[r] \arrow[d] & M_2 \otimes_A Q \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}
$$

with exact horizontal sequences. By the observation above all the three vertical columns are exact. If M is flat, then the left most map in the middle horizontal row is injective. Applying the Snake Lemma to rows two and three in the diagram gives that $M_2 \otimes_A N \longrightarrow M_2 \otimes_A F$ is injective. Hence M_2 is flat by Proposition (6.4.5). Similar argument shows that M_1 also is flat. Thus if $M = M_1 \oplus M_2$ is flat, we get that it's summands are also flat. The converse is immediate consequence of the fact that tensor products commute with direct sum (Proposition (6.1.2).

□

6.4.7 Corollary. *Any projective module is flat.*

Proof. Let P be a projective module, and write P as a quotient of a free module $F \longrightarrow P$. By Lemma (5.3.9) that surjection splits, and we have that $F = P \oplus K$, where K is the kernel of $F \longrightarrow P$. Since F is free we have by Example (6.4.3) that F is flat. Then, as just seen we get that each of its summands, and in particular the projective module P, is flat.

□

6.4.8 Corollary. *A finitely generated* **Z***-module is flat if and only if it is free.*

Proof. By the Fundamental Theorem for finitely generated abelian groups we have that a finitely generated module M can be written as

$$M = \oplus_{i=1}^{n} \mathbf{Z} \oplus_{j=1}^{m} \mathbf{Z}/(n_j),$$

for some integers $n_1, \ldots, n_m$, all strictly greater than one. By the Proposition M is flat if and only if all of its summands are flat. A torsion module $\mathbf{Z}/(n)$ with $n \geq 1$ is not flat, as seen in Example (6.4.4). Thus the only summands that are flat are summands with $\mathbf{Z}$. Such a module will be free by definition.

□

6.5 Tor

We fix an A-module M. For any A-module N we let

$$\longrightarrow P_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} P_0 \longrightarrow N \longrightarrow 0$$

be a projective resolution of N. Tensoring that resolution with our fixed module M we get the complex

$$\longrightarrow P_n \otimes_A M \longrightarrow \cdots \longrightarrow P_0 \otimes_A M \longrightarrow 0.$$

If $P_\bullet$ was the resolution of N, we let $(P_\bullet \otimes_A M)$ denote the associated complex we get by tensoring with M. For each positive integer $n \geq 0$, we have the homology modules

$$H_n(P_\bullet \otimes_A M) = \mathrm{Ker}(d_n \otimes 1)/\mathrm{Im}(d_{n+1} \otimes 1).$$

6.5.1 Proposition. *The homology modules defined above are independent of the projective resolution. These homology modules are denoted*

$$\mathrm{Tor}_n^A(N, M) := H_n(P_\bullet \otimes_A M).$$

Proof. The proof, which is similar to the proof of (5.3.5), is left to the reader.

□

6.5.2 Remark. Note that $\mathrm{Tor}_0^A(N, M) = N \otimes_A M$.

6.5.3 Proposition. *Let* $0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$ *be a short exact sequence of A-modules. Then there is, for any A-module M, a long exact sequence*

$$\begin{tikzcd}
 & & \cdots \arrow[r] & \mathrm{Tor}^A_{n+1}(N_3, M) \arrow[dlll] \\
\mathrm{Tor}^A_n(N_1, M) \arrow[r] & \mathrm{Tor}^A_n(N_2, M) \arrow[r] & \mathrm{Tor}^A_n(N_3, M) \arrow[dll] & \\
\mathrm{Tor}^A_{n-1}(N_1, M) \arrow[r] & \cdots & &
\end{tikzcd}$$

Proof. By Lemma (5.3.8) we can find projective resolutions of the modules N_1, N_2 and N_3, that together make a short exact sequence in degree n;

$$0 \longrightarrow P_n \longrightarrow Q_n \longrightarrow R_n \longrightarrow 0.$$

We have furthermore that the sequence splits by Lemma (5.3.9), hence $Q_n = P_n \oplus R_n$. Thus, when tensoring the resolutions with M, we get complexes in the vertical direction, but the short exactness in each degree n is preserved. The long exact sequence of homology modules then follows from Theorem (4.1.3) and Proposition (6.5.1).

□

6.5.4 Proposition. *Let M be an A-module. The following assertions are equivalent*

(1) The module M is flat.

(2) We have $\mathrm{Tor}^A_1(N, M) = 0$ for all A-modules N.

(3) We have $\mathrm{Tor}^A_n(N, M) = 0$ for all N, all $n > 0$.

Proof. Clearly (3) implies (2). By the long exact sequence (6.5.3) it follows that (2) implies (1). We will show that (1) implies (3). Assume that M is flat, and let N be a module. Let $P_\bullet$ be a projective resolution of N. By taking out kernels and cokernels, one can break up any long exact sequence into many short exact sequences. Since M is flat, tensoring with M will preserve short exact sequences. It follows that M also preserves any long exact sequences. Thus $P_\bullet \otimes_A M$ is not only a complex, it is exact. Hence all the positive homology modules are zero, and we have shown that (1) implies (3).

□

6.6 Universal Coefficient Theorem

6.6.1 Theorem. *Assume that the ring A is either the integers $A = \mathbf{Z}$, or that A is a field. Let M be a finitely generated A-module, and let $F_\bullet$ be a complex where in each degree the modules are free and of finite rank. The following identity holds,*

$$H_n(F_\bullet \otimes_A M) = H_n(F_\bullet) \otimes_A M \bigoplus \mathrm{Tor}_1^A(H_{n-1}(F_\bullet), M).$$

Proof. Let $\{F_n, d_n\}$ be a complex with free and finitely generated modules F_n. Let $Z_n = \mathrm{Ker}(d_n)$. We have $Z_n \otimes_A M \subseteq \ker(d_n \otimes 1)$, and the isomorphism of Exercise (1.8.6) gives the short exact sequence

$$Z_n \otimes_A M/\mathrm{Im}(d_{n+1} \otimes 1) \subset \ker(d_n \otimes 1)/\mathrm{Im}(d_{n+1} \otimes 1) \to \ker(d_n \otimes 1)/Z_n \otimes_A M \to 0 \tag{6.2}$$

Note that by definition the middle term is $H_n(F_\bullet \otimes_A M)$. We will prove the theorem by showing that this short exact sequence splits, giving us that the middle term is the direct summand of the two others, and then identify these two modules. Let us begin by recalling that the sequence

$$0 \longrightarrow Z_n \longrightarrow F_n \longrightarrow B_{n-1} \longrightarrow 0 \tag{6.3}$$

is short exact, where $B_{n-1} = \mathrm{im}(d_n)$. As $Z_n \subseteq F_n$ and $B_{n-1} \subseteq F_{n-1}$ are submodules of a free module, these two modules themselves are free (Proposition (5.2.2)). It then follows from Lemma (5.3.9) that the above sequence splits, so $F_n = Z_n \oplus B_{n-1}$. We then get that $Z_n \otimes_A M$ is a summand of $F_n \otimes_A M$, and then also that $Z_n \otimes_A M$ is a summand of $\ker(d_n \otimes 1)$. We then obtain that the sequence (6.2) splits. To identify the outer terms of the sequence (6.2), consider the exact sequence

$$0 \longrightarrow Z_n \longrightarrow F_n \xrightarrow{d_n} Z_{n-1} \longrightarrow H_{n-1}(F_\bullet) \longrightarrow 0.$$

Which yields a free, hence projective, resolution of $H_{n-1}(F_\bullet)$. Consequently the homology of the complex

$$Z_n \otimes_A M \xrightarrow{j} F_n \otimes_A M \xrightarrow{d_n \otimes 1} Z_{n-1} \otimes_A M$$

computes the torsion modules $\mathrm{Tor}_i^A(H_{n-1}(F_\bullet), M)$. Note that the splitting of the sequence (6.3) implies that $Z_n \otimes_A M$ is a summand of $F_n \otimes_A M$, and consequently that the homomorphism j is injective. We then have

$$\mathrm{Tor}_0^A(H_{n-1}(F_\bullet), M) = Z_{n-1} \otimes_A M/\mathrm{Im}(d_n \otimes 1)$$
$$\mathrm{Tor}_1^A(H_{n-1}(F_\bullet), M) = \ker(d_n \otimes 1)/Z_n \otimes_A M.$$

As $H_{n-1}(F_\bullet) \otimes_A M$ is the zero'th Tor-module $\mathrm{Tor}_0^A(H_{n-1}(F_\bullet), M)$ we have identified the ends of the sequence (6.2) in the desired way, proving our claim. □

6.6.2 Remark. The assumption that the complex $F_\bullet$ consists of modules that are free and of *finite* rank in each degree, can be relaxed. In fact, what we used is that any submodule of a free **Z**-module is free, and that holds without the finiteness assumption.

6.7 Exercises

6.7.1. Let M, N and P be A-modules. Show that we have a natural isomorphism

$$\mathrm{Hom}_A(M \otimes_A N, P) = \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)).$$

6.7.2. Let E be a free A-module of finite rank, and let E^* denote its dual. Show that for any A-module M we have a natural isomorphism

$$\mathrm{Hom}_A(M, E) = \mathrm{Hom}_A(M \otimes_A E^*, A).$$

6.7.3. Let M be an A-module. We define the tensor algebra $T_A(M)$ in the following way. Let $T^0M = A$. For each integer $n > 0$ we let $T^n(M) = M \otimes_A \cdots \otimes_A M$ be the tensor product of M taken n-times. So $T^1(M) = M, T^2(M) = M \otimes_A M$ and so on. Let

$$T_A(M) = \oplus_{n \geq 0} T^n(M),$$

and show that the A-module $T_A(M)$ has a natural product structure by

$$(x_1 \otimes \cdots \otimes x_n) \cdot (y_1 \otimes \cdots \otimes y_m) := x_1 \otimes \cdots \otimes x_n \otimes y_1 \cdots \otimes y_m.$$

6.7.4. Let M be an A-module, and let $T_A(M)$ be the tensor algebra (6.7.3). Let $I \subseteq T_A(M)$ be the two-sided ideal generated by all tensors of the form $x \otimes y - y \otimes x$, for all $x, y \in M$. Let $S_A(M) := T_A(M)/I$ denote the graded commutative unital quotient algebra, which we refer to as the symmetric algebra.

(1) Show that for any A-algebra (commutative) B there is a bijection between A-module homomorphisms $M \longrightarrow B$ and A-algebra homomorphisms $S_A(M) \longrightarrow B$.

(2) Show that when M is a free module of rank n, then $S_A(M) = A[x_1, \ldots, x_n]$ the polynomial algebra in n-variables over A.

6.7.5. Let $A = \mathbf{Z}$ be the integers. The rank of a finitely generated A-module M is defined as the $\mathbf{Q}$-vector space dimension

$$\mathrm{rk}(M) = \dim_{\mathbf{Q}}(M \otimes_{\mathbf{Z}} \mathbf{Q}).$$

Show that for a free and finitely generated module F the rank defined here coincides with the usual rank defined (1.4.11).

6.7.6. Let $P_n \longrightarrow \cdots \longrightarrow P_0$ be a complex of finitely generated $\mathbf{Z}$-modules. Show that

$$\sum_{i=0}^{n}(-1)^i \mathrm{rk}(P_i) = \sum_{i=0}^{n}(-1)^i \mathrm{rk} H_i(C_\bullet),$$

where $H_i(C_\bullet)$ is the i'th homology module of the complex.

6.7.7. Assume that we have a short exact sequence $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ of finitely generated $\mathbf{Z}$-modules. Show that the Euler-Poincare characteristic (5.5.3) is additive;

$$\chi(M_2) = \chi(M_1) + \chi(M_3).$$

6.7.8. Show that an A-module M is flat if and only if the natural map $M \otimes_A I \longrightarrow A$ is injective, for all ideals $I \subseteq A$.

6.7.9. Compute $\mathrm{Tor}^n_{\mathbf{Z}/(8)}(\mathbf{Z}/(4), \mathbf{Z}/(4))$.

6.7.10. Let I and J be two ideals in a ring A. Show that the ideal $I + J$ (see Exercise (1.8.5)) consisting of all A-linear combinations of elements in I and J, equals the smallest ideal containing both I and J. That is, $I + J = \cap I'$, where the intersection is taken over all ideal $I' \subseteq A$, with $I \subseteq I'$ and $J \subseteq I'$. Deduce then that

$$A/I \otimes_A A/J = A/(I + J).$$

6.7.11. Let I and J be two ideals in a ring A. Show that

$$\begin{aligned}
&\mathrm{Tor}^n_A(A/I, A/J) = \mathrm{Tor}^{n-2}_A(I, J) \quad (n > 2)\\
&\mathrm{Tor}^2_A(A/I, A/J) = \mathrm{Ker}(I \otimes_A J \longrightarrow IJ).\\
&\mathrm{Tor}^1_A(A/I, A/J) = I \cap J/(IJ).
\end{aligned}$$

6.7.12. Show that if M and N are flat A-modules, then so is $M \otimes_A N$.

6.7.13. Given a short exact sequence $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$ of A-modules. Show that if M_1 and M are flat, then so is M_2.

Chapter 7

Topological spaces

The aim of this chapter is to introduce the main objects we are going to study: topological spaces and continuos maps between them. We will discuss several examples: Euclidean spaces and its subspaces, simplices, quotients, projective spaces, and manifolds. We will also discuss compactness. Homotopy relation is the subject of the last part of this chapter.

7.1 Topological spaces and continuous maps

7.1.1 Definition. *A topological space is a set X together with a collection $\mathcal{T}$ of subsets of X which satisfies the following properties:*

(1) X and $\emptyset$ belong to $\mathcal{T}$.

(2) If U and V belong to $\mathcal{T}$, then so does the intersection $U \cap V$.

(3) If, for $s \in S$, U_s belong to $\mathcal{T}$, then so does the union $\bigcup_{s\in S} U_s$.

Let $(X, \mathcal{T})$ be a topological space. The collection $\mathcal{T}$ is called the **topology** of X and the members of $\mathcal{T}$ are called **open** subsets of X. A topological space $(X, \mathcal{T})$ will be often denoted simply by X, in which case the collection of open subsets $\mathcal{T}$ is assumed to be known. A subset $D \subset X$ is called **closed** if the complement $X \setminus D$ is open.

7.1.2 Excercise. Let $(X, \mathcal{T})$ be a topological space. Show that:

(1) X and $\emptyset$ are closed subset of X.

(2) If D and E are closed subset of X, then so is $D \cup E$.

(3) If, for $s \in S$, D_s is closed in X, then so is $\cap_{s\in S} D_s$.

7.1.3 Example. Let X be the set consisting of just one point. Such a set has a unique topology consisting of all the subsets of X. This topological space is denoted by Δ^0 or D^0 or $\mathbf{R}^0$ and is called the point.

7.1.4 Definition. *Let X and Y be topological spaces. A function $f : X \longrightarrow Y$ is called continuous if, for any open subset $V \subset Y$, the pre-image $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open in X.*

We will often use the term **map** for a continuous function between topological spaces. Thus any map is a function, but not all functions are maps, only those that are continuous. This notion of continuity is essential for understanding topological spaces. Maps will be used to compare spaces.

7.1.5 Excercise. Show that a function $f : X \longrightarrow Y$ is continuous if and only if, for any closed subset $D \subset Y$, the pre-image $f^{-1}(D) = \{x \in X \mid f(x) \in D\}$ is closed in X.

7.1.6 Excercise. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be continuous functions between topological spaces. Show that the composition $gf : X \longrightarrow Z$ is also continuous.

7.1.7 Definition. *A continuous function $f : X \longrightarrow Y$ is called an isomorphism if there is a continuous function $g : Y \to X$ such that $fg = id_Y$ and $gf = id_X$. Two spaces X and Y are said to be isomorphic if there is an isomorphism $f : X \longrightarrow Y$.*

Note that an isomorphism of topological spaces is a one to one and onto function (such functions are also called bijections), as it has an inverse function. However even if a continuous function is a bijection (one to one and onto), so it has an inverse, the inverse may fail to be continuous. Thus to be an isomorphism it is not enough to be a continuous bijection. To be an isomorphism, in addition to having the inverse, this inverse has to be continuous.

7.1.8 Excercise. Let S be a set containing at least two distinct elements. Consider the following two collections of subsets of S: $\mathcal{T} = \{\emptyset, S\}$ and $\mathcal{D} = \{\text{all subsets of } S\}$. Show that $(S, \mathcal{T})$ and $(S, \mathcal{D})$ are topological spaces. Prove that $\mathrm{id} : S \longrightarrow S$ is a continuous function between $(S, \mathcal{D})$ and $(S, \mathcal{T})$, but it is not a continuous function between $(S, \mathcal{T})$ and $(S, \mathcal{D})$. Conclude that $\mathrm{id} : S \longrightarrow S$ is not an isomorphism between $(S, \mathcal{D})$ and $(S, \mathcal{T})$. Are $(S, \mathcal{D})$ and $(S, \mathcal{T})$ isomorphic?

Constructing new spaces

7.2 Subspaces

Let X be a topological space and $Y \subset X$ be a subset. Define $\mathcal{U}$ to be the collection of subsets of Y which consist of intersections $Y \cap U$ where U is an open subset in X. We call the collection $\mathcal{U}$ the subspace topology on Y.

7.2.1 Excercise. Let X be a topological space and $Y \subset X$ be a subset. Show that Y with the subspace topology is a topological space.

7.2.2 Excercise. Let X be a topological space and $Z \subset Y \subset X$ be subsets. Consider Y as a topological subspace of X. The set Z can be then considered as a subspace of Y and a subspace of X. Show that these two subspace topologies on Z are the same.

We will often use the above exercises to construct new topological spaces. We will define first an "ambient" topological space and then consider its subspaces as the new topological spaces.

7.2.3 Excercise. Let Y and X be topological spaces and $Z \subset X$ be a topological subspace. Show that the inclusion $i : Z \subset X$ is continuous. Show also that a function $f : Y \longrightarrow Z$ is continuous if and only if the composition $if : Y \longrightarrow X$ is continuous.

7.3 Disjoint unions

Let X and Y be topological spaces. Consider the disjoint union $X \coprod Y$ and the collection $\mathcal{U}$ of subsets $U \subset X \coprod Y$ such that $U \cap X$ is open in X and $U \cap Y$ is open in Y. We call the collection $\mathcal{U}$ the disjoint union topology on $X \coprod Y$.

More generally, for a collection of topological spaces $\{X_i\}_{i \in I}$, consider the disjoint union $\coprod_{i \in I} X_i$ and the collection $\mathcal{U}$ of subsets $U \subset \coprod_{i \in I} X_i$ such that $U \cap X_i$ is open for any $i \in I$. We call the collection $\mathcal{U}$ the disjoint union topology on $\coprod_{i \in I} X_i$.

7.3.1 Excercise. Let Y and X be topological spaces. Show $X \coprod Y$, with the disjoint union topology, is a topological space.

7.3.2 Excercise. Let X,Y, and Z be topological spaces. Show that a function $f : X \coprod Y \longrightarrow Z$ is continuous if and only if the compositions of f and the inclusions $\mathrm{in}_1 : X \subset X \coprod Y$ and $\mathrm{in}_2 : Y \subset X \coprod Y$ are both continuous.

7.3.3 Excercise. A space of the form $\coprod_I \Delta^0$ is called discreet. Show that any subset of a discreet space is open. Show that any function out of a discreet space is continuous. Show that any function into Δ^0 is continuous.

7.4 Products

Let X and Y be topological spaces. Consider the product $X \times Y$ and the collection $\mathcal{U}$ of subsets $U \subset X \times Y$ such that, for any point $(x, y) \in U$, there are open subsets $x \in U_1 \subset X$ and $y \in U_2 \subset Y$ such that $U_1 \times U_2 \subset U$. We call the collection $\mathcal{U}$ the product topology on $X \times Y$.

7.4.1 Excercise. Let X and Y be topological spaces. Show $X \times Y$ with the product topology is a topological space.

7.4.2 Excercise. Let X, Y, and Z be topological spaces. Show that a function $f : Z \to X \times Y$ is continuous if and only if the compositions of f with projections $\mathrm{pr}_1 : X \times Y \longrightarrow X$ and $\mathrm{pr}_2 : X \times Y \longrightarrow Y$ are both continuous.

7.5 Quotients

Let X be a topological space, Y a set, and $f : X \longrightarrow Y$ a function of sets. The function f can be used to define a topology on Y. Let $\mathcal{U}$ be the collection of subset $U \subset Y$ for which $f^{-1}(U)$ is open in X. The collection $\mathcal{U}$ is called the quotient topology on Y induced by f. We will also call Y, with the quotient topology, a quotient space of X.

7.5.1 Excercise. Let X be a topological space. Show that Y with the quotient topology induced by $f : X \longrightarrow Y$ is a topological space and the function $f : X \longrightarrow Y$ is continuous.

An important property of a quotient space is that it is easy to verify if a function from such a space is continuous:

7.5.2 Proposition. *Let Y be the topological space given by the quotient topology induced by $f : X \longrightarrow Y$. A function $g : Y \to Z$ is continuous if and only if the composition $gf : X \longrightarrow Z$ is continuous.*

Proof. Composition of continuous functions is continuous (Exercise 7.1.6). Thus if g is continuous, then so is gf. This shows one implication.

Assume now that gf is continuous. We need to show that g is continuous. Let $U \subset Z$ be an open subset. As gf is continuous, the subset $f^{-1}(g^{-1}(U))$ is open in X. By definition of the quotient topology, $g^{-1}(U)$ is then open in Y and we can conclude that g is also continuous. $\square$

7.6 Compact spaces

The aim of this section is to introduce an important class of topological spaces: compact spaces. Their key property is that continuous bijections between them are isomorphisms. We start with:

7.6.1 Definition.

(1) *A topological space X is called* **Hausdorff** *if, for any two distinct paints $x_1, x_2 \in X$, there are open subsets $x_1 \in U_1 \subset X$ and $x_2 \in U_2 \subset X$ whose intersection $U_1 \cap U_2$ is empty.*

(2) *A topological space X is* **compact** *if it is Hausdorff and, for any collection of open subsets $\{U_i \subset X\}_{i \in I}$ for which $\bigcup_{i\in I} U_i = X$, there is a finite sequence $i_1, i_2, \ldots, i_k$ such that:*

$$U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_k} = X$$

The collection of open subsets $\{U_i \subset X\}_{i \in I}$ such that $\bigcup_{i\in I} U_i = X$ is called an open cover, or simply a cover, of X. A space X is compact if it is Hausdorff and any cover of X has a finite subcover.

7.6.2 Excercise. Let X and Y be spaces. The disjoint union $X \coprod Y$ is compact if and only if both X and Y are compact.

7.6.3 Excercise. Let $\{X_i\}_{i\in I}$ be a collection of compact spaces. Show that $\coprod_{i\in I} X_i$ is compact if and only if I is a finite set.

7.6.4 Excercise. Let X and Y be spaces. The product $X \times Y$ is compact if and only if both X and Y are compact.

Here are some fundamental properties of compact spaces:

7.6.5 Proposition.

(1) *Let X be a Hausdorff space. If a subspace $Y \subset X$ is compact, then it is a closed subset of X.*

(2) *Let X be a compact space. A subspace $Y \subset X$ is compact if and only if it is closed.*

Proof. (1): We need to show that $X \setminus Y$ is open. For that it is enough to prove that, for any point $x \notin Y$, there is an open set $x \in U \subset X$ such that the intersection $U \cap Y$ is empty. Since X is Hausdorff, for any $y \in Y$, we can find two open subsets $x \in U_y \subset X$ and $y \in V_y \subset X$ such that $U_y \cap V_y = \emptyset$.

It is then clear that $Y \subset \bigcup_{y \in Y} V_y$. Since Y is compact, we can then find finitely many points $y_1, \ldots, y_n$, such that $Y \subset V_{y_1} \cup \cdots \cup V_{y_n}$. It then follows that an open subset $U = U_{y_1} \cap \cdots \cap U_{y_n}$ has empty intersection with Y.

(2): If Y is compact then it is closed by statement (1).

Let Y be a closed subset of X and $Y = \bigcup_{i \in I}(U_i \cap Y)$, where U_i is open in X. Then $X = (X \setminus Y) \cup \bigcup_{i \in I} U_i$. Since X is compact we can find then a finite sequence $i_1, i_2, \ldots, i_k$ such that:

$$X = (X \setminus Y) \cup U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_k}$$

It then follows that $Y = (U_{i_1} \cap Y) \cup (U_{i_2} \cap Y) \cup \cdots \cup (U_{i_k} \cap Y)$. To show that Y is compact, it remains to show that it is Hausdorff. Let y_1 and y_2 be two distinct points in Y. Since X is Hausdorff, there are two open subsets $y_1 \in U_1 \subset X$ and $y_2 \in U_2 \subset X$ whose intersection $U_1 \cap U_2$ is empty. It then follows that the intersection of $U_1 \cap Y$ and $U_2 \cap Y$ is also empty and Y is Hausdorff. $\square$

An important property of compactness is that it is preserved by continuous functions:

7.6.6 Proposition.

(1) *Let $f : X \longrightarrow Y$ be a continuous function and Y a Hausdorff space. If $D \subset X$ is compact, then $f(D)$ is compact and closed in Y.*

(2) *Assume that X is compact and Y Hausdorff. If $f : X \longrightarrow Y$ is a map which is an onto function, then Y is also compact.*

(3) *Assume that X is compact and Y is Hausdorff. If $f : X \longrightarrow Y$ is a continuos bijection (one to one and onto), then f is an isomorphism.*

Proof. (1): Since Y is Hausdorff, then so is $f(D)$. Let $f(D) \subset \bigcup_{i \in I} U_i$. Since D is compact and $D \subset \bigcup_{i \in I} f^{-1}(U_i)$, there is a finite sequence $i_1, \ldots, i_k$ such that:

$$D \subset f^{-1}(U_{i_1}) \cup f^{-1}(U_{i_2}) \cup \cdots \cup f^{-1}(U_{i_k})$$

Consequently $f(D) \subset U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_k}$. We can conclude that $f(D)$ is compact. By Proposition 7.6.5.(1), $f(D)$ is also closed in Y, as Y is assumed to be Hausdorff.

(2): This is a consequence of statement (1).

(3): Since f is a bijection, there is an inverse function $g : Y \longrightarrow X$, such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$. We need to show that g is continuous. According to Exercise 7.1.5 it is enough to prove that $g^{-1}(D)$ is closed in Y, for any closed subset $D \subset X$. Note however that $g^{-1}(D) = f(D)$. Thus we need to show that $f(D)$ is closed. This follows from the following sequence of implications. Since D is closed and X is compact, D is also compact by Proposition 7.6.5.(2). As Y is Hausdorff, $f(D)$ is then compact by statement (1). We can then use again Proposition 7.6.5.(1) to conclude that $f(D)$ is closed. $\square$

According to statement (3) of the above proposition, to show that compact spaces X and Y are isomorphic, it is enough to construct a continuous bijection $f : X \longrightarrow Y$. The inverse of f would be then necessarily continuous. This is one of the key advantages of compact spaces and we will use it often. Another property of compact spaces often used is:

7.6.7 Proposition. *Let X be a Hausdorff space and $Y \subset X$ and $Z \subset X$ be two disjoint ($Y \cap Z = \emptyset$) subspaces which are compact. Then there are open sets $Y \subset U \subset X$ and $Z \subset V \subset X$ such that $U \cap V = \emptyset$.*

Proof. Let us fix a point $y \in Y$. For any $z \in Z$ let us choose open subsets $y \in U_{y,z} \subset X$ and $z \in V_{y,z} \subset X$ such that $U_{y,z} \cap V_{y,z} = \emptyset$. This can be done since X is Hausdorff. Clearly $Z \subset \bigcup_{z\in Z} V_{y,z}$. Since Z is compact, there is a finite sequence $z_1, z_2, \ldots, z_k$ such that:

$$Z \subset V_{y,z_1} \cup V_{y,z_2} \cup \cdots \cup V_{y,z_k}$$

Define $U_y := U_{y,z_1} \cap Y_{y,z_2} \cap \cdots \cap U_{y,z_k}$ and $V_y := V_{y,z_1} \cup V_{y,z_2} \cup \cdots \cup V_{y,z_k}$. Then U_y and V_y are disjoint open subsets of X such that $y \in U_y$ and $Z \subset V_y$. Consider such subsets for all $y \in Y$. It is clear that $Y \subset \bigcup_{y\in Y} U_y$. Since Y is compact, there is a finite sequence $y_1, y_2, \ldots, y_n$ such that:

$$Y \subset U_{y_1} \cup U_{y_2} \cup \cdots \cup U_{y_k}$$

Define $U := U_{y_1} \cup U_{y_2} \cup \cdots \cup U_{y_k}$ and $V := V_{y_1} \cap V_{y_2} \cap \cdots \cap V_{y_k}$. The subsets U and V satisfy the requirements of the propositions. $\square$

7.7 Euclidean spaces

For $n > 0$, let $\mathbf{R}^n$ be the set of n-tuples of real numbers. If $n = 0$, we define $\mathbf{R}^0$ to be the one point set $\{0\}$. Recall that $|x| = \sqrt{x_1^2 + \cdots x_n^2}$, if $n > 0$ and $|0| = 0$ if $n = 0$. Let $a \in \mathbf{R}^n$ and $r \in \mathbf{R}$. The following subsets in $\mathbf{R}^n$ are called respectively the sphere, the disc, and the open ball with center in a and radius r:

$$S(a,r) = \{x \in \mathbf{R}^n \mid |x-a| = r\}$$
$$D(a,r) = \{x \in \mathbf{R}^n \mid |x-a| \leq r\}$$
$$B(a,r) = \{x \in \mathbf{R}^n \mid |x-a| < r\}$$

We can now define a topology on $\mathbf{R}^n$. A subset $U \subset \mathbf{R}^n$ is called open if, for any point $a \in U$, there is a number $\epsilon > 0$, such that the open ball $B(a,\epsilon)$ is included entirely in U. The collection of such open subsets of $\mathbf{R}^n$ is called the Euclidean topology and $\mathbf{R}^n$, with this choice of open subsets, is called the n-dimensional Euclidean space. We will not consider any other topology on $\mathbf{R}^n$. From now on the symbol $\mathbf{R}^n$ denotes the n-dimensional Euclidean space.

7.7.1 Excercise. Show that $\mathbf{R}^n$ with the above choice of open subsets is a Hausdorff topological space.

7.7.2 Excercise. Show that the product of Euclidean topologies on $\mathbf{R}^n \times \mathbf{R}^m$ is the same as the Euclidean topology on $\mathbf{R}^{n+m}$.

7.7.3 Excercise. Consider $\mathbf{R}^n$ as a subset of $\mathbf{R}^{n+m}$ consisting of $n+m$-tuples of real numbers whose last m-coordinates are 0. Show that the Euclidean topology on $\mathbf{R}^n$ is the same as the subspace topology of the Euclidean topology on $\mathbf{R}^{n+m}$.

We can now use the Euclidean space $\mathbf{R}^n$ to consider its subspaces. In this way we can get a lot of new examples of topological spaces. Here are some of the ones we will often use:

$$n\text{–dimensional disc:} \quad D^n := \{x \in \mathbf{R}^n \mid |x| \leq 1\}$$
$$n\text{–dimensional open disc:} \quad B^n := \{x \in \mathbf{R}^n \mid |x| < 1\}$$
$$(n-1)\text{–dimensional sphere:} \quad S^{n-1} := \{x \in \mathbf{R}^n \mid |x| = 1\}$$
$$\text{unit interval:} \quad I := \{x \in \mathbf{R} \mid 0 \leq x \leq 1\}$$
$$n\text{–dimensional simplex:} \quad \Delta^n := \{x \in \mathbf{R}^{n+1} \mid x_0 + \cdots + x_n = 1\}$$

For example $D^0 = \mathbf{R}^0 = \Delta^0$ is the one point space. The space $S^0 \subset \mathbf{R}$ consists of two points $\{-1, 1\}$ and it is then isomorphic to the disjoint union $D^0 \coprod D^0$.

Let $0 \leq i \leq n$. A point in Δ^n whose all coordinates are 0 except the i-th coordinate, which has to be 1, is called the i-th vertex of Δ^n and is denoted by v_i.

7.7.4 Excercise. Let $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^m$ are subspaces. Show that a function $f : X \longrightarrow Y$ is continuous if and only if, for any $a \in X$ and any $\epsilon > 0$, there is $\delta > 0$ such that, when $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

7.7.5 Excercise. Show that, for any $a \in \mathbf{R}^n$ and any $r > 0$, the spaces $\mathbf{R}^n$ and $B(a, r)$ are isomorphic.

7.7.6 Excercise. Show that, for any $a, b \in \mathbf{R}^n$ and any $r, s > 0$, the spaces $D(a, r)$ and $D(b, s)$ are isomorphic.

7.7.7 Excercise. Show that D^n and Δ^n are isomorphic spaces.

7.7.8 Theorem. *A subspace X of the Euclidean space $\mathbf{R}^n$ is compact if and only if, as a subset of R^n, it is closed and bounded (it lies in a ball $B(a, r)$ for some r).*

It follows from the above theorem that S^{n-1}, D^n, and Δ^n are compact spaces. It turns out that the Euclidean spaces $\mathbf{R}^n$, for different n, are not isomorphic. Can you see how to show it? Part of this course is to develop methods that can be used to distinguish topological spaces. The general idea to show that, for $m \neq n$, $\mathbf{R}^m$ is not isomorphic to $\mathbf{R}^n$, is to find a property of topological spaces which is preserved by isomorphisms and such that $\mathbf{R}^m$ satisfies this property and $\mathbf{R}^n$ does not. We will use homology to identify such a property.

7.8 Simplicial operators

For $0 \leq i \leq n+1$, define $d_i : \Delta^n \longrightarrow \Delta^{n+1}$ to be the function given by:

$$d_i(x_0, \ldots x_n) = (x_0, \ldots x_{i-1}, 0, x_i, \ldots x_n) \in \Delta^{n+1}$$

For $0 \leq i \leq n$, define $s_i : \Delta^{n+1} \to \Delta^n$ to be the function given by:

$$s_i(x_0, \ldots x_{n+1}) = (x_0, \ldots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \ldots, x_{n+1})$$

One can use Exercise 7.7.4 to see that d_i and s_i are continuous.

The symbols $d_j d_i$, $d_j s_i$, $s_j d_i$ and $s_j s_i$ denote respectively the following compositions:

$$\Delta^n \xrightarrow{d_i} \Delta^{n+1} \xrightarrow{d_j} \Delta^{n+2}$$

$$\Delta^{n+1} \xrightarrow{s_i} \Delta^n \xrightarrow{d_j} \Delta^{n+1}$$

$$\Delta^n \xrightarrow{d_i} \Delta^{n+1} \xrightarrow{s_j} \Delta^n$$

$$\Delta^{n+2} \xrightarrow{s_i} \Delta^{n+1} \xrightarrow{s_j} \Delta^n$$

7.8.1 Excercise. Show that the following identities hold:

(1) If $j > i$, then $d_j d_i = d_i d_{j-1}$.

(2) If $j > i$, then $s_j d_i = d_i s_{j-1}$.

(3) $s_i d_i = \mathrm{id}$.

(4) $s_i d_{i+1} = \mathrm{id}$.

(5) If $j - 1 < i$, then $s_j d_i = d_{i-1} s_j$.

(6) If $i > j$, then $s_j s_i = s_{i-1} d_j$.

7.9 Cell attachments

Let X be a topological space and $\alpha : S^{n-1} \longrightarrow X$ be a map. Consider the space $X \coprod D^n$ with the disjoint union topology. Consider further the set $X \coprod B^n$ and a function $f : X \coprod D^n \longrightarrow X \coprod B^n$ given by:

$$f(x) = \begin{cases} x & \text{if } x \in X \coprod B^n \\ \alpha(x) & \text{if } x \in S^{n-1} \subset D^n \end{cases}$$

The topological space consisting of the set $X \coprod B^n$ together with the quotient topology given by f is denoted by $X \cup_\alpha D^n$. We say that $X \cup_\alpha D^n$ is constructed out of X by attaching n-dimensional cell D^n along $\alpha : S^{n-1} \longrightarrow X$.

7.9.1 Excercise. Let $\alpha : S^{n-1} \longrightarrow X$ be a map. Show that the composition of the quotient map $f : X \coprod D^n \longrightarrow X \cup_\alpha D^n$ and the inclusion $X \coprod B^n \subset X \coprod D^n$, induced by the identity $\mathrm{id} : X \longrightarrow X$ and the inclusion $B^n \subset D^n$, is a continuous bijection. Is it an isomorphism?

7.9.2 Proposition. *If X is Hausdorff, then so is $X \cup_\alpha D^n$, for any map $\alpha : S^{n-1} \to X$.*

Proof. Let y_1 and y_2 be two distinct points in $X \cup_\alpha D^n$. There are 3 cases. First y_1 and y_2 are in X. Since X is Hausdorff, then there are two disjoint open subsets $y_1 \in U_1 \subset X$ and $y_2 \in U_2 \subset X$. The subsets $\alpha^{-1}(U_1)$ and $\alpha^{-1}(U_2)$ are open in S^{n-1}. Define:

$$V_i := \{x \in D^n \mid |x| > 1/2 \text{ and } x/|x| \in \alpha^{-1}(U_i)\} \subset D^n$$

Note that V_1 and V_2 are open and disjoint subsets of D^n. Finally set $W_i := U_i \cup (V_i \setminus \alpha^{-1}(U_i) \subset X \coprod B^n$. Note that $f^{-1}(W_i) = U_i \cup V_i \subset X \coprod D^n$. Thus

W_i is open in $X \cup_\alpha D^n$. The subsets W_1 and W_2 are also disjoint and contain respectively y_1 and y_2.

Let $y_1 \in X$ and $y_2 \in B^n$. Define $W_1 := X \cup \{x \in B^n \mid |x| > (1+|y_2|)/2\}$ and $W_2 := \{x \in B^n \mid |x| < (1+|y_2|)/2\}$. Note that $f^{-1}(W_1) = X \cup \{x \in D^n \mid |x| > (1+|y_2|)/2\}$ and $f^{-1}(W_2) = \{x \in D^n \mid |x| < (1+|y_2|)/2\}$. These are open subsets. The sets W_i are then also open. They are disjoint and contain respectively y_1 and y_2.

Let y_1 and y_2 be two distinct points in B^n. Since B^n is Hausdorff, there are open disjoint subsets $y_1 \in W_1 \subset B^n$ and $y_2 \in W_2 \subset B^n$. The subset W_i is also open in Y. □

7.9.3 Proposition. *If X is compact, then so is $X \cup_\alpha D^n$, for any map $\alpha : S^{n-1} \to X$.*

Proof. By previous proposition $X \cup_\alpha D^n$ is Hausdorff. Since $X \coprod D^n$ is compact, according to Proposition 7.6.6.(2), the space $X \cup_\alpha D^n$, as the image of $f : X \coprod D^n \longrightarrow X \cup_\alpha D^n$, is also compact. □

7.9.4 Example. Consider the one point space D^0. Let $\alpha : S^{n-1} \longrightarrow D^0$ be the unique map. The space $D^0 \cup_\alpha D^n$ is isomorphic to S^n. To construct the isomorphism consider a function $g : D^0 \coprod B^n \longrightarrow S^n$ defined as follows:

$$g(x) = \begin{cases} (0, \cdots, 0, -1) & \text{if } x \in D^0 \\ (2\sqrt{\frac{1-|x|}{|x|}}x, 1 - 2|x|) & \text{if } x \in B^n \setminus \{(0, \cdots, 0)\} \\ (0, \cdots, 0, 1) & \text{if } x = (0, \cdots, 0) \in B^n \end{cases}$$

We claim that g is a bijection and that the composition of the quotient function $f : D^0 \coprod D^n \longrightarrow D^0 \coprod B^n$ and g is a continuous function $gf : D^0 \coprod D^n \longrightarrow S^n$. To see this we need to show that the restriction of gf to the components D^0 and D^n are continuous. This is true for the first restriction since all functions out of D^0 are continuous. The other restriction is given by the formula:

$$D^n \ni x \mapsto \begin{cases} (2\sqrt{\frac{1-|x|}{|x|}}x, 1 - 2|x|) \in S^n & \text{if } x \in D^n \setminus \{(0, \cdots, 0)\} \\ (0, \cdots, 0, 1) \in S^n & \text{if } x = (0, \cdots, 0) \in D^n \end{cases}$$

whose continuity can be checked using 7.7.4. Thus the function g defines a continuous map, denoted by the same symbol, $g : D^0 \cup_\alpha D^n \longrightarrow S^n$. As both spaces $D^0 \cup_\alpha D^n$ and S^n are compact, this map g must be then an isomorphism.

7.10 Real projective spaces

Consider the Euclidean space $\mathbf{R}^{n+1}$ ($n \geq 0$). The symbol $\mathbf{RP}^n$ denotes the set of 1-dimensional $\mathbf{R}$-vector subspaces of $\mathbf{R}^{n+1}$. Such subspaces are also called lines in $\mathbf{R}^{n+1}$. For example since $\mathbf{R}$ is 1-dimensional $\mathbf{R}$-vector space, it has only one 1-dimensional $\mathbf{R}$-vector subspace, and hence $\mathbf{RP}^0$ is just a point.

Define $\pi : S^n \longrightarrow \mathbf{RP}^n$ to be the function that assigns to a vector $v \in S^n \subset \mathbf{R}^{n+1}$ the $\mathbf{R}$-linear subspace generated by v. Explicitly $\pi(v) := \{rv \mid r \in \mathbf{R}\}$.

7.10.1 Definition. *The set $\mathbf{RP}^n$ together with the quotient topology induced by $\pi : S^n \longrightarrow \mathbf{RP}^n$ is called the n-dimensional real projective space.*

In the rest of this section we will identify the projective spaces as spaces build by attaching cells. For that we need:

7.10.2 Proposition. $\mathbf{RP}^n$ *is a compact space.*

Proof. Since S^n is compact, according to Proposition 7.6.6.(2), to show that $\mathbf{RP}^n$ is compact it is enough to prove that it is Hausdorff. Let L_1 and L_2 be two distinct points in $\mathbf{RP}^n$, i.e., two distinct 1-dimensional $\mathbf{R}$-linear subspaces in $\mathbf{R}^{n+1}$. Let v_1 and v_2 be two points in S^n which generate the lines L_1 and L_2 respectively. Let $r = \min\{|v_1 - v_2|, |v_1 + v_2|\}$. Define U_1 to be the subset of $\mathbf{PR}^n$ of all the lines which are generated by vectors v such that $\min\{|v_1 - v|, |v_1 + v|\} < r/2$. Define U_2 to be subset of $\mathbf{RP}^n$ of all the lines which are generated by vectors v such that $\min\{|v_2 - v|, |v_2 + v|\} < r/2$. It is clear that the subsets U_1 and U_2 are disjoint and $L_1 \in U_1$ and $L_2 \in U_2$. We claim that these sets are also open. Note that:

$$\pi^{-1}(U_i) = \{w \in S^n \mid |v_i - w| < r/2\} \cup \{w \in S^n \mid |v_i + w| < r/2\}$$

Since it is an open subset in S^n, U_i is open in $\mathbf{RP}^n$. $\square$

We can use the map $\pi : S^n \longrightarrow \mathbf{RP}^n$ to attach a cell and build a new topological space $\mathbf{RP}^n \cup_\pi D^{n+1}$.

7.10.3 Proposition. *The space $\mathbf{RP}^n \cup_\pi D^{n+1}$ is isomorphic to $\mathbf{RP}^{n+1}$.*

Proof. Since both spaces $\mathbf{RP}^n \cup_\pi D^{n+1}$ and $\mathbf{RP}^{n+1}$ are compact, to show that they are isomorphic we need to construct a continuous bijection between them. Let us denote by $e : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^{n+2}$, respectively $e : S^n \longrightarrow S^{n+1}$, the functions which assigns to an element $x \in \mathbf{R}^{n+1}$ the element $(x, 0) \in \mathbf{R}^{n+2}$. Note that both of these functions are continuous. Define $i : \mathbf{RP}^n \longrightarrow \mathbf{RP}^{n+1}$

to be a function that assigns to a line $L \subset \mathbf{R}^{n+1}$, the line in $e(L) \subset \mathbf{R}^{n+2}$. Note that there is a commutative diagram:

$$\begin{tikzcd} S^n \arrow[r, "e"] \arrow[d, "\pi"'] & S^{n+1} \arrow[d, "\pi"] \\ \mathbf{RP}^n \arrow[r, "i"] & \mathbf{RP}^{n+1} \end{tikzcd}$$

Since the composition $i\pi = \pi e$ is continuous, then i is also continuous. The map $i : \mathbf{RP}^n \longrightarrow \mathbf{RP}^{n+1}$ is called the standard inclusion.
Define a function $g : \mathbf{RP}^n \coprod B^{n+1} \to \mathbf{RP}^{n+1}$ as follows:

$$g(x) = \begin{cases} i(x) & \text{if } x \in \mathbf{RP}^n \\ \text{the line generated by } (x, \sqrt{1-|x|^2}) \in \mathbf{R}^{n+2} & \text{if } x \in B^{n+1} \end{cases}$$

Note that the composition of the function g with the quotient map $f : \mathbf{RP}^n \coprod D^{n+1} \longrightarrow \mathbf{RP}^n \cup_\pi D^{n+1}$ is continuous. Thus g defines a continuous map $g : \mathbf{RP}^n \cup_\pi D^{n+1} \longrightarrow \mathbf{RP}^{n+1}$. Note that this map is a bijection. Since the spaces are compact we can conclude that this map is an isomorphism. □

7.10.4 Example. Let $n = 0$. In this case $\mathbf{RP}^0$ is just a point and as a topological space it is isomorphic to D^0. Thus there is only one map $\pi : S^0 \longrightarrow \mathbf{RP}^0$. The space $\mathbf{RP}^1$ is then isomorphic to $D^0 \cup_\pi D^1$, which by Example 7.9.4, is isomorphic to S^1. It follows then that $\mathbf{RP}^1$ is isomorphic to S^1. We would like to identify the map $\pi : S^1 \longrightarrow \mathbf{RP}^1 = S^1$. Note that this map sands the elements x and $-x$ to the same point. If we think about S^1 as a subset of the complex numbers $\mathbf{C}$ of length 1, then the multiplication map $S^1 \ni z \mapsto z^2 \in S^1$ also sends x and $-x$ to the same point. One can then check directly that the map $\pi : S^1 \longrightarrow \mathbf{RP}^1 = S^1$ is given by this multiplication map.

7.11 Topological manifolds

7.11.1 Definition. *Let $n \geq 0$. A topological space X is called an n-dimensional manifold if it is a Hausdorff space and, for any point $x \in X$, there is an open subset $x \in U \subset X$ which is isomorphic to $\mathbf{R}^n$.*

We will be interested mostly in topological manifolds which are compact. The most basic are:

7.11.2 Proposition. *Let X be a compact topological manifold of dimension 0. Then X is isomorphic to $D^0 \coprod D^0 \coprod \cdots \coprod D^0$ (a finite disjoint union of points).*

Proof. Since, for any point $x \in X$, there is an open subset $x \in U \subset X$ which is isomorphic to $\mathbf{R}^0$ we can conclude that $\{x\} \subset X$ is an open subset of X. As $X = \cup_{x \in X}\{x\}$ and X is compact, we must have $X = \{x_1, x_2, \dots, x_k\}$. The map $\coprod_{i=1}^{k} D^0 \longrightarrow X$ which sends the i-th D^0 to the point x_i is then a continuous bijection. Since both spaces are compact this bijection has to be then an isomorphism. □

Can a manifold have different dimensions? This question is directly related to the question if $\mathbf{R}^n$ and $\mathbf{R}^m$ can be isomorphic for different n and m. We will use homology to answer this question.

7.11.3 Proposition. *Spaces S^n and $\mathbf{RP}^n$ are compact manifolds of dimension n.*

Proof. Let us choose points $x = (x_1, \dots, x_{n+1}) \in S^n$ and $\pi(x) \in \mathbf{RP}^n$. Consider the subsets $U := \{y \in S^n \mid |y-x| < 1\}$ and $D := \{y \in S^n \mid |y-x| \leq 1\}$. Note that U is open in S^n. Moreover its image $\pi(U)$ is open in $\mathbf{RP}^n$, as $\pi^{-1}\pi(U) = U \cup (-U)$ is open in S^n. The subset D is closed in S^n and hence it is a compact space. Since the map $\pi : D \to \pi(D)$ is a continuous bijection it has to be an isomorphism. Its restriction $\pi : U \longrightarrow \pi(U)$ is therefore also an isomorphism. Thus to show the proposition it is enough to prove that U is isomorphic to $\mathbf{R}^n$. We are going to show that U is isomorphic to B^n. One can then use Exercise 7.7.5 to get an isomorphism between U and $\mathbf{R}^{n-1}$. Define $f : U \longrightarrow B^{n-1}$ and $g : B^{n-1} \longrightarrow U$ by the following formulas:

$$f(y_1, \dots, y_n) = (y_1 - x_1, \dots, y_{n-1} - x_{n-1})$$

$$g(z_1, \dots, z_{n-1}) =$$

$$= \left(z_1 + x_1, \dots, z_{n-1} + x_{n-1}, \sqrt{1 - (z_1 + x_1)^2 - \cdots - (z_{n-1} + x_{n-1})^2}\right)$$

It is clear that f and g are continuous. Since the compositions fg and gf are identities, these morphisms are isomorphisms. □

7.12 Homotopy relation

Recall that I denotes the unit interval $[0, 1] \subset \mathbf{R}$. We are going to use it to define a relation on continuous maps with the same domain and range:

7.12.1 Definition. *A map $f : X \longrightarrow Y$ is homotopic to $g : X \longrightarrow Y$ if there is a map $S : X \times I \longrightarrow Y$ such that, for any $x \in X$, $S(x, 0) = f(x)$ and $S(x, 1) = g(x)$. Any such map S is called a homotopy between f and g.*

Homotopy is an equivalence relation on the set of continuous maps between X and Y and is preserved by compositions:

7.12.2 Proposition. *(1) If $f : X \longrightarrow Y$ is homotopic to $g : X \longrightarrow Y$ then g is homotopic to f (symmetry of the homotopy relation).*

(2) If $f : X \longrightarrow Y$ is homotopic to $g : X \longrightarrow Y$ and $g : X \longrightarrow Y$ is homotopic to $h : X \longrightarrow Y$, then f is homotopic to h (transitivity of the homotopy relation).

(3) If $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ are homotopic, then, for any $h : Y \longrightarrow Z$, so are the compositions $hf : X \longrightarrow Z$ and $hg : X \longrightarrow Z$.

(4) If $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ are homotopic, then, for any $h : Z \longrightarrow X$, so are the compositions $fh : Z \longrightarrow Y$ and $gh : Z \longrightarrow Y$.

Proof. (1): Let $S : X \times I \longrightarrow Y$ be a homotopy between f and g. Define $S' : X \times I \longrightarrow Y$ as $S'(x, t) = S(x, 1 - t)$. Note that S' is a homotopy between g and f.

(2): Let $S : X \times I \longrightarrow Y$ be a homotopy between f and g and $S' : X \times I \longrightarrow Y$ be a homotopy between g and h. Define $S'' : X \times I \longrightarrow Y$ by the formula:

$$S''(x, t) = \begin{cases} S(x, 2t) & \text{if } 0 \leq t \leq 1/2 \\ S'(x, 2t - 1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

Note that $S''(x, 0) = f(x)$ and $S''(x, 1) = h(x)$. Thus, if continuous, S'' would be a homotopy between f and h. To see that S'' is continuous consider the following compositions:

$$X \times [0, 1/2] \xrightarrow{\alpha} X \times I \xrightarrow{S} Y$$

$$X \times [1/2, 1] \xrightarrow{\beta} X \times I \xrightarrow{S'} Y$$

where $\alpha(x, t) = (x, 2t)$ and $\beta(x, t) = (x, 2t - 1)$. These composition are clearly continuous. It follows that, if D is closed in Y, then so are $(S\alpha)^{-1}(D) \subset X \times [0, 1/2] \subset X \times I$ and $(S'\beta)^{-1}(D) \subset X \times [1/2, 1] \subset X \times I$. The sum $(S\alpha)^{-1}(D) \cup (S'\beta)^{-1}(D)$ is then also closed in $X \times I$. Note however that this sum coincide with $(S'')^{-1}(D)$. The function S'' is therefore continuous.

(3): If $S : X \times I \longrightarrow Y$ is a homotopy between f and g, then hS is a homotopy between hf and hg.

(4): Let $S : X \times I \longrightarrow Y$ be a homotopy between f and g. Define $S' : Z \times I \longrightarrow Y$ by the formula $S'(z,t) = S(h(x),t)$. This is a homotopy between fh and gh. $\square$

The fundamental example of homotopic maps are the inclusions $\mathrm{in}_0 : X \longrightarrow X \times I$ and $\mathrm{in}_1 : X \longrightarrow X \times I$, where $\mathrm{in}_0(x) = (x,0)$ and $\mathrm{in}_1(x) = (x,1)$.

Homotopy relation can be used to define:

7.12.3 Definition. *A map $f : X \longrightarrow Y$ is called a homotopy equivalence if there is a map $g : Y \longrightarrow X$ such that the compositions fg and gf are homotopic respectively to the identity maps $id : Y \longrightarrow Y$ and $id : X \longrightarrow X$.*

Two spaces X and Y are said to be homotopy equivalent if there is a homotopy equivalence $f : X \longrightarrow Y$.

A space is called contractible if it is homotopy equivalent to the one point space D^0.

The homotopy equivalence relation on spaces is an equivalence relation. It is a weaker relation than an isomorphism. Two isomorphic spaces are clearly homotopy equivalent.

7.12.4 Proposition. *(1) If $D \subset \mathbf{R}^n$ is non-empty and convex (the interval between any two points in D is subset of D), then D is contractible.*

(2) The spaces $\mathbf{R}^n$, D^n, Δ^n, and B^n are contractible.

Proof. (1): Let us choose a point $x \in D$. Define $f : D^0 \longrightarrow D$ to be given by $f(0) = x$ and $g : D \longrightarrow D^0$ to be the unique map. Clearly $gf = \mathrm{id}$. We need to show that the composition $fg : D \longrightarrow D$ is homotopic to $\mathrm{id} : D \longrightarrow D$. Define $S : D \times I \longrightarrow D$ by the formula: $S(y,t) = tx + (1-t)y$. It is well define since D is convex. Note that $s(y,0) = y$ and $s(y,1) = fg$. Thus S is a homotopy between id and fg.

(2): This follows from statement (1) as all these spaces are convex. $\square$

7.12.5 Proposition. *(1) Let $D \subset \mathbf{R}^n$ be convex and $x \in D$ a point for which there is $r > 0$ such that $B(x,r) \subset D$. Then the space $D \setminus \{x\}$ is homotopy equivalent to S^{n-1}.*

(2) Let $n > 0$. The spaces $\mathbf{R}^n \setminus \{0\}$, $D^n \setminus \{0\}$, $\Delta^n \setminus \{\big(1/(n+1), \ldots, 1/(n+1)\big)\}$, and $B^n \setminus \{0\}$ are homotopy equivalent to S^{n-1}.

Proof. Since D is convex, the space $S(x, r/2)$ is a subspace of D. This space $S(x, r/2)$ is isomorphic to S^{n-1}. Thus to show the statement it is enough to show that $D \setminus \{x\}$ is homotopy equivalent to $S(x, r/2)$. Set a map $f : S(x, r/2) \longrightarrow D \setminus \{x\}$ to be the inclusion and $g : D \setminus \{x\} \longrightarrow S(x, r/2)$ to be defined by the formula $g(y) = x + \frac{r(y-x)}{2|y-x|}$. It is straight forward to check that gf is $\mathrm{id} : S(x, r/2) \longrightarrow S(x, r/2)$. We need to show that fg is homotopic to $\mathrm{id} : D \setminus \{x\} \longrightarrow D \setminus \{x\}$. Define $H : (D \setminus \{x\}) \times I \longrightarrow D \setminus \{x\}$:

$$H(y, t) := ty + (1-t)\big(x + \frac{r(y-x)}{2|y-x|}\big)$$

Note that $H(y, 1) = \mathrm{id}$ and $H(y, 0) = fg(y)$. □

7.13 $\pi_0(X)$

Let X be a topological space. Note that maps $f : D^0 \longrightarrow X$ can be identified with elements of X. Such a map is determined by where it sends the only point of D^0. In this way we can think about elements of X as maps from the one point space D^0 to X. The homotopy relation on such maps can be then rephrased in terms of elements of X as follows: two points $x_0, x_1 \in X$ are homotopic if there is a map $\alpha : I \longrightarrow X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Such continuous maps are called paths between x_0 and x_1. To have a language to describe this particular situation we are going to use the following definition:

7.13.1 Definition. *Two points $x_0 \in X$ and $x_1 \in X$ are said to be in the same path component if there is a path $\alpha : I \longrightarrow X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$.*

Since homotopy relation on maps is an equivalence relation (see Proposition 7.12.2), we get that being in the same path component is also an equivalence relation on the set of elements of X. We can then consider the equivalence classes of this relation and define:

7.13.2 Definition. *The set of equivalence classes of the relation "being in the same path component" on the set of points of X is denoted by $\pi_0(X)$.*

A space is called path connected if $\pi_0(X)$ consists of one point, i.e., if all pairs of points in X are in the same path component.

7.13.3 Excercise. Show that, for any $n \geq 0$, Δ^n is path connected.

We are going to denote elements in $\pi_0(X)$ by $[x]$, where $x \in X$ is a point in the given equivalence class. For such an element $[x] \in \pi_0(X)$, let us denote by $X_{[x]}$ the subspace of X consisting of all points in X that are in the same path component as x. These subspaces of X are called path connected components of X. Note that these subspaces are path connected, i.e., $\pi_0(X_{[x]})$ is the one point set.

7.13.4 Excercise. Show that if Y is path connected, then for any map $f : Y \longrightarrow X$, there is $x \in X$ such that $f(Y) \subset X_{[x]}$.

Chapter 8

Homology

In this chapter we are going to define and discuss homology of topological spaces. We are going to illustrate how to use this invariant to distinguish between topological spaces. In particular we are going to show that, for different n's, the spaces $\mathbf{R}^n$'s are not isomorphic. We will present explicit calculations of the homology of spheres and projective spaces. We will also study the effect on homology groups of various standard maps.

8.1 Singular homology

Let X be a topological space. How can we study X? We are going to do that by comparing X to simplices of various dimensions. This will be done by studying sets of maps between Δ^n and X.

8.1.1 Definition.

- *A singular simplex of dimension n in X is a continuous map $\sigma : \Delta^n \longrightarrow X$.*
- *The set of singular simplexes of dimension n in X is denoted by $\Delta_n(X)$.*
- *The symbol $S_n(X)$ denotes the free abelian group generated by the set $\Delta_n(X)$.*

For example, the set $\Delta_0(X)$, of maps from Δ^0 to X, can be identified with the set of points of X.

Any elements of the group $S_n(X)$ can be written uniquely as a sum:

$$a_1\sigma_1 + a_1\sigma_2 + \cdots + a_k\sigma_k$$

where, for all l, $a_l \in \mathbf{Z}$ is an integer and $\sigma_l : \Delta^n \longrightarrow X$ is a singular simplex in X.

To get interesting information about X it is not enough however to study just singular simplices $\Delta^n \longrightarrow X$. We need to also understand how simplicial operators act on singular simplices. By precomposing with the simplicial operator $d_i : \Delta^n \longrightarrow \Delta^{n+1}$ (see 7.8), we get a map of sets and a group homomorphism which we denote by the same symbol:

$$d_i : \Delta_{n+1}(X) \longrightarrow \Delta_n(X) \qquad (\sigma : \Delta^{n+1} \longrightarrow X) \mapsto (\sigma d_i : \Delta^n \longrightarrow X)$$

$$d_i : S_{n+1}(X) \longrightarrow S_n(X) \qquad \sum_l a_l \sigma_l \mapsto \sum_l a_l(\sigma_l d_i)$$

8.1.2 Excercise. Show that if $j > i$, then the following compositions of maps of sets are the same (compare with 7.8.1):

$$\Delta_{n+2}(X) \xrightarrow{d_i} \Delta_{n+1}(X) \xrightarrow{d_j} \Delta_n(X)$$

$$\Delta_{n+2}(X) \xrightarrow{d_{j-1}} \Delta_{n+1}(X) \xrightarrow{d_i} \Delta_n(X)$$

Conclude that the following compositions describe the same group homomorphisms:

$$S_{n+2}(X) \xrightarrow{d_i} S_{n+1}(X) \xrightarrow{d_j} S_n(X)$$

$$S_{n+2}(X) \xrightarrow{d_{j-1}} S_{n+1}(X) \xrightarrow{d_i} S_n(X)$$

8.1.3 Example. Let $x_0, \dots, x_n$ be $n+1$ points in a convex subspace X of $\mathbf{R}^m$. These points determine a singular simplex $\sigma : \Delta^n \longrightarrow \mathbf{X}$ given by $\sigma(t_0, \dots, t_n) = t_0 x_0 + \cdots t_n x_n$. We will also use the symbol $(x_0, \dots, x_n)$ to denote this singular simplex. Such singular simplices in X are called **linear**.

If σ is a linear simplex in X, then so is $d_i(\sigma)$. Explicitly, if $\sigma = (x_0, \dots, x_n)$, then $d_i(\sigma) = (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. We will also denote the linear simplex $d_i(\sigma)$ by $(x_0, \dots, \hat{x}_i, \dots, x_n)$, where the symbol $\hat{x}_i$ indicates the omitted component.

For example, consider the convex subspace $\Delta^n \times I \subset \mathbf{R}^{n+1} \times \mathbf{R}$. For each $0 \leq k \leq n$, the following set of $n+2$ points $\Delta^n \times I$ determines a linear simplex of dimension $n+1$, which we denote by $\tau_k : \Delta^{n+1} \longrightarrow \Delta^n \times I$:

$$\big((v_0, 0), \dots, (v_k, 0), (v_k, 1), \dots, (v_n, 1)\big)$$

where v_i denotes the i-th vertex of Δ^n, i.e. the point whose i-th coordinate is 1 and all other coordinates are 0.

8.1.4 Example. Since Δ^0 is the one point space, 0-dimensional singular simplices in X can be identified with elements $x \in X$. The group $\Delta_0(X)$ is then the free abelian group generated by elements of X. An element of $S_0(X)$ is a combination of elements of X with integer coefficients:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

where $a_i \in \mathbf{Z}$ and $x_i \in X$.
Since Δ^1 is isomorphic to the interval I, we can think about 1-dimensional singular simplices in X also as maps $\alpha : I \longrightarrow X$. Under this identification, if a 1-dimensional simplex σ in X corresponds to $\alpha : I \longrightarrow X$, then $d_0(\sigma)$ corresponds to the end $\alpha(1)$ of the path α. In the same way $d_1(\sigma)$ corresponds to the beginning $\alpha(0)$ of the path α.

8.1.5 Definition. *Let $n \geq 0$. Define $\partial_{n+1} : S_{n+1}(X) \longrightarrow S_n(X)$ to be given by:*

$$\partial_{n+1}(\sigma) = \sum_{i=0}^{n+1}(-1)^i d_i(\sigma)$$

8.1.6 Proposition. *Let $n \geq 0$. The following composition is the zero homomorphism:*

$$S_{n+2}(X) \xrightarrow{\partial_{n+2}} S_{n+1}(X) \xrightarrow{\partial_{n+1}} S_n(X)$$

Proof.

$$\partial_{n+1}\partial_{n+2}(x) = \sum_{j=0}^{n+1}(-1)^j d_j\big(\sum_{i=0}^{n+2}(-1)^i d_i(x)\big) =$$

$$= \sum_{j>i}(-1)^{i+j} d_j d_i(x) + \sum_{j\leq i}(-1)^{i+j} d_j d_i(x) =$$

use Exercise 8.1.2 to get:

$$= \sum_{j>i}(-1)^{i+j} d_i d_{j-1}(x) + \sum_{j\leq i}(-1)^{i+j} d_j d_i(x) =$$

set $j = k+1$ in the first sum to get:

$$= \sum_{k\geq i}(-1)^{i+k+1} d_i d_k(x) + \sum_{j\leq i}(-1)^{i+j} d_j d_i(x) = 0$$

□

We can now use the above proposition to define a chain complex of abelian groups:

8.1.7 Definition. *Let X be a topological space. The following chain complex is called the singular chain complex of X and is denoted by $S_*(X)$:*

$$\cdots \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_3} S_2(X) \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X)$$

Let G be an abelian group and X a topological space. The n-th homology of the complex $S_*(X) \otimes G$ is called the n-th homology of X with coefficients in G and is denoted by $H_n(X, G)$.
In the case G is the group of integers $\mathbf{Z}$, then $H_n(X, \mathbf{Z})$ is simply denoted by $H_n(X)$ and called the integral homology of X.

8.1.8 Example. Note that $S_*(D^0)$ is given by the complex:

$$\cdots \xrightarrow{\mathrm{id}} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{\mathrm{id}} \mathbf{Z} \xrightarrow{0} \cdots \xrightarrow{\mathrm{id}} \mathbf{Z} \xrightarrow{0} \mathbf{Z}$$

It follows that:

$$H_n(D^0, G) = \begin{cases} G & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

8.1.9 Excercise. Show that there are isomorphisms:

$$S_*(\coprod_{i=1}^{r} D^0) = \bigoplus_{i=1}^{r} S_*(D^0) \qquad H_n(\coprod_{i=1}^{r} D^0, G) = \bigoplus_{i=1}^{r} H_n(D^0, G)$$

8.1.10 Excercise. As in Example 8.1.4 we think about 1-dimensional singular simplices in X as paths $\sigma : I \longrightarrow X$. For such a path, let us denote by $\sigma' : I \longrightarrow X$ the opposite path, i.e., by definition, $\sigma'(t) := \sigma(1-t)$. Show that $\sigma + \sigma'$ is a boundary in $S_1(X)$ (it is in the image of $\partial_2 : S_2(X) \longrightarrow S_1(X)$).

8.1.11 Excercise. Let $\sigma : I \longrightarrow X$ and $\tau : I \longrightarrow X$ be two paths such that $\sigma(1) = \tau(0)$. Define $\sigma * \tau : I \longrightarrow X$ to be the concatenation of σ and τ, explicitly:

$$\sigma * \tau(t) := \begin{cases} \sigma(2t) & \text{if} \quad 0 \leq t \leq 1/2 \\ \tau(2-2t) & \text{if} \quad 1/2 \leq t \leq 1 \end{cases}$$

Show that $\sigma + \tau - \sigma * \tau$ is a boundary in $S_1(X)$.

The rest of these notes is devoted to setting up tools to calculate homology of more complicated spaces than just discreet spaces. We start with discussing:

8.2 Homology and maps

To compare spaces one uses maps. Maps can be also used to compare the homology groups of spaces. Let $f : X \longrightarrow Y$ be a continuous function. By composing with f we get a function between the sets of singular simplices:

$$\Delta_n(X) \ni (\sigma : \Delta^n \longrightarrow X) \mapsto (f\sigma : \Delta^n \longrightarrow Y) \in \Delta_n(Y)$$

This function is denoted by $\Delta_n(f) : \Delta_n(X) \longrightarrow \Delta_n(Y)$. The induced homomorphism on the groups of singular simplices is denoted by:

$$S_n(f) : S_n(X) \longrightarrow S_n(Y)$$

It maps an element $a_1\sigma_1+\cdots+a_k\sigma_k$ in $S_n(X)$ to an element $a_1 f\sigma_1+\cdots+a_k f\sigma_k$ in $S_n(Y)$.

8.2.1 Excercise. Show that the following diagrams of abelian groups and homomorphisms commute:

$$\begin{tikzcd} S_{n+1}(X) \arrow[r, "S_{n+1}(f)"] \arrow[d, "d_i"] & S_{n+1}(Y) \arrow[d, "d_i"] \\ S_n(X) \arrow[r, "S_n(f)"] & S_n(Y) \end{tikzcd} \qquad \begin{tikzcd} S_{n+1}(X) \arrow[r, "S_{n+1}(f)"] \arrow[d, "\partial_{n+1}"] & S_{n+1}(Y) \arrow[d, "\partial_{n+1}"] \\ S_n(X) \arrow[r, "S_n(f)"] & S_n(Y) \end{tikzcd}$$

8.2.2 Excercise. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be continuous maps. Show that the homomorphism $S_n(gf) : S_*(X) \longrightarrow S_n(Z)$ is given by the composition of $S_n(f) : S_n(X) \longrightarrow S_n(Y)$ and $S_n(g) : S_n(Y) \longrightarrow S_n(Z)$. Moreover show that $S_n(\mathrm{id}) : S_n(X) \longrightarrow S_n(X)$ is the identity homomorphism. Conclude that if $f : X \longrightarrow Y$ is an isomorphism of topological spaces, then $S_n(f) : S_n(X) \longrightarrow S_n(Y)$ is an isomorphism of abelian groups.

From the above exercises it follows that f induces a map of chain complexes $S_*(f) : S_*(X) \longrightarrow S_*(Y)$ and hence a homomorphism of homology groups $H_n(f, G) : H_n(X, G) \longrightarrow H_n(Y, G)$. These induced homomorphisms are compatible with compositions:

$$S_*(fg) = S_*(f)S_*(g) \quad S_*(\mathrm{id}) = \mathrm{id}$$

$$H_n(fg, G) = H_n(f, G)H_n(g, G) \quad H_n(\mathrm{id}, G) = \mathrm{id}$$

Moreover if $f : X \longrightarrow Y$ is an isomorphism, then so are $S_*(f)$ and $H_n(f, G)$.

8.3 Reduced homology

For any space X, there is a unique map $p : X \longrightarrow D^0$. This map induces then a homomorphism of chain complexes $S_*(p) : S_*(X) \longrightarrow S_*(D^0)$. We use this homomorphism to define:

8.3.1 Definition. *Let X be a topological space. The kernel of $S_*(p) : S_*(X) \longrightarrow S_*(D^0)$ is denoted by $\tilde{S}_*(X)$ and called the complex of reduced singular simplices of X.*

For a group G, the n-th homology of $\tilde{S}_*(X) \otimes G$ is denoted by $\tilde{H}_n(X, G)$ and called the n-th reduced homology of X with coefficients in G. If $G = \mathbf{Z}$ is the group of integers, then $\tilde{H}_n(X, \mathbf{Z})$ is simply denoted by $\tilde{H}_n(X)$.

For example, $\tilde{S}_*(D^0)$ is the zero complex and consequently $\tilde{H}_n(D^0, G) = 0$ for any $n \geq 0$ and any group G.

For any continuous map $f : X \longrightarrow Y$, the following triangle commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\ & D^0 & \end{tikzcd}$$

By applying S_* to this triangle, we get a commutative triangle of homomorphisms of chain complexes:

$$\begin{tikzcd} S_*(X) \arrow[rr, "S_*(f)"] \arrow[dr, "S_*(p)"'] & & S_*(Y) \arrow[dl, "S_*(p)"] \\ & S_*(D^0) & \end{tikzcd}$$

Commutativity of this diagram implies that $S_n(f) : S_n(X) \longrightarrow S_n(Y)$ takes elements of $\tilde{S}_n(X)$ into elements of $\tilde{S}_n(Y)$. Thus, for any such continuous map, restriction of $S_*(f)$ to the subcomplex $\tilde{S}_*(X)$ defines a chain map which is denoted by:

$$\tilde{S}_*(f) : \tilde{S}_*(X) \longrightarrow \tilde{S}_*(Y)$$

By tensoring with a group G we get a yet another homomorphism of chain complexes:

$$\tilde{S}_*(f) \otimes G : \tilde{S}_*(X) \otimes G \longrightarrow \tilde{S}_*(Y) \otimes G$$

The induced homomorphism on the n-th homology is denoted by:

$$\tilde{H}_n(f, G) : \tilde{H}_n(X, G) \longrightarrow \tilde{H}_n(Y, G)$$

8.3.2 Excercise. Show that if $f : X \longrightarrow Y$ is an isomorphism then so is $\tilde{H}_n(f, G)$, for any $n \geq 0$ and any group G.

Recall that $S_*(D^0)$ can be identified with the complex (see Example 8.1.8):

$$\cdots \xrightarrow{\text{id}} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{\text{id}} \mathbf{Z} \xrightarrow{0} \cdots \xrightarrow{\text{id}} \mathbf{Z} \xrightarrow{0} \mathbf{Z}$$

Moreover, with this identification, the homomorphism $S_n(p) : S_n(X) \longrightarrow \mathbf{Z}$ is given by:

$$S_n(p)(a_1\sigma_1 + \cdots + a_k\sigma_k) = a_1 + \cdots + a_k$$

Thus $\tilde{S}_n(X)$ is the subgroup of $S_n(X)$ consisting of these sums $a_1\sigma_1 + \cdots + a_k\sigma_k$ for which $a_1 + \cdots + a_k = 0$.

In the case X is a nonempty space, then there is a map $D^0 \longrightarrow X$ whose composition with $p : X \longrightarrow D^0$ is the identity. It follows that after applying S_* to these maps we get a commuting diagram:

$$\begin{tikzcd} S_*(D^0) \arrow[r] \arrow[rr, bend right, "\text{id}"'] & S_*(X) \arrow[r, "S_*(p)"] & S_*(D^0) \end{tikzcd}$$

It then follows that the chain complex $S_*(X)$ is isomorphic to the direct sum of complexes $S_*(D_0) \oplus \tilde{S}_*(X)$.

8.3.3 Proposition. *Let X be a non empty space and G be a group.*

(1) $H_0(X, G)$ is isomorphic to $G \oplus \tilde{H}_0(X, G)$.

(2) For $n > 0$, the homomorphism $\tilde{H}_n(X, G) \longrightarrow H_n(X, G)$, induced by the inclusion $\tilde{S}_*(X) \subset S_*(X)$, is an isomorphism.

Proof. Since X is not empty, the following is a split exact sequence of chain complexes:

$$0 \longrightarrow \tilde{S}_*(X) \subset S_*(X) \xrightarrow{S_*(p)} S_*(D^0) \longrightarrow 0$$

Thus by tensoring with G we get also a split exact sequence of chain complexes:

$$0 \longrightarrow \tilde{S}_*(X) \otimes G \subset S_*(X) \otimes G \xrightarrow{S_*(p)\otimes G} S_*(D^0) \otimes G \longrightarrow 0$$

Such a short exact sequence of chain complexes leads to a long exact sequence of its homologies:

$$
\begin{array}{l}
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots \longrightarrow \\
\longrightarrow \tilde{H}_1(X,G) \longrightarrow H_1(X,G) \xrightarrow{H_1(p,G)} H_1(D^0,G) \longrightarrow \\
\longrightarrow \tilde{H}_0(X,G) \longrightarrow H_0(X,G) \xrightarrow{H_0(p,G)} H_0(D^0,G) \longrightarrow 0
\end{array}
$$

The proposition is now a direct consequence of the fact that: $H_0(D^0, G) = G$ and $H_n(D^0, G) = 0$ for $n > 0$. □

8.4 $H_0(X, G)$

The aim of this section is to explain what information about a topological space is encoded in its 0'th homology. Note first that, by definition, $H_0(X)$ fits into an exact sequence:

$$S_1(X) \xrightarrow{\partial_1} S_0(X) \longrightarrow H_0(X) \longrightarrow 0$$

Thus an element of $H_0(X)$ can be represented by an element in $S_0(X)$, i.e., a sum $a_1x_1 + \cdots + a_kx_k$ where $a_i \in \mathbf{Z}$ and $x_i \in X$. Two such sums give the same element in $H_0(X)$ if their difference is of the form $\partial_1(\sigma)$ for some $\sigma \in S_1(X)$.

As tensoring is right exact, for any group G, the above exact sequence leads to an exact sequence:

$$S_1(X) \otimes G \xrightarrow{\partial_1} S_0(X) \otimes G \longrightarrow H_0(X) \otimes G \longrightarrow 0$$

It follows that:

8.4.1 Proposition. *The groups $H_0(X, G)$ and $H_0(X) \otimes G$ are isomorphic.*

A topological space consists of a set X together with a topology (a choice of open subsets). We used this topology to define an equivalence relation on the set X: "being in the same path component" (see Definition 7.13.1).

If 0 and 1 dimensional singular simplices in X are identified with respectively elements of X and paths $\alpha : I \longrightarrow X$ in X, as explained in Example 8.1.4, then the relation of being in the same path component can be reformulated as follows:

8.4.2 Proposition. *Two points $x_0 \in X$ and $x_1 \in X$ are in the same path component if and only if there is a singular simplex $\sigma : \Delta^1 \longrightarrow X$ such that $d_0(\sigma) = x_1$ and $d_1(\sigma) = x_0$.*

Recall that for $x \in X$, the symbol $X_{[x]}$ denotes the subspace of X consisting of all the points in X that are in the same path component as x.

8.4.3 Proposition. *For any G and $n \geq 0$, there are isomorphisms:*

$$\coprod_{[x]\in\pi_0(X)} \Delta_n(X_{[x]}) = \Delta_n(X)$$

$$\bigoplus_{[x]\in\pi_0(X)} S_*(X_{[x]}) = S_*(X) \qquad \bigoplus_{[x]\in\pi_0(X)} H_n(X_{[x]}, G) = H_n(X, G)$$

Proof. Since Δ^n is path connected, any singular simplex $\sigma : \Delta^n \longrightarrow X$ maps Δ^n into one of the path connected components of X. The first isomorphism is then induced by the inclusions $X_{[x]} \subset X$. The second and the third isomorphisms are consequences of the first one. $\square$

8.4.4 Proposition. *If X is non empty and path connected, then, for the unique map $p : X \longrightarrow D^0$, the homomorphism $H_0(p, G) : H_0(X, G) \longrightarrow H_0(D^0, G)$ is an isomorphism.*

Proof. Choose a point $x \in X$ and consider the composition:

$$D^0 \xrightarrow{\pi} X \xrightarrow{p} D^0$$

where π sends D^0 to the chosen point x. Since this composition is the identity, then so is the induced composition on the homology:

$$H_0(D^0) \xrightarrow{H_0(\pi)} H_0(X) \xrightarrow{H_0(p)} H_0(D^0)$$

It follows that $H_0(\pi)$ is a monomorphism. We are going to show that it is also an epimorphism. Let $a_1x_1 + \cdots + a_kx_k$ be an element in $S_0(X)$ that represents a given homology class in $H_0(X)$. Since X is path connected, for any i, there is a path $\alpha_i : I \longrightarrow X$ such that $\alpha_i(0) = x$ and $\alpha_i(1) = x_i$. Consider the following element in $S_1(X)$:

$$\sigma := a_1\alpha_1 + \cdots + a_k\alpha_k$$

Note that $\partial_1(\sigma) = a_1x_1 + \cdots + a_kx_k - (a_1 + \cdots + a_k)x$. Consequently the element $a_1x_1+\cdots+a_kx_k$ differs from $(a_1+\cdots+a_k)x$ by en element of the form $\partial_1(\sigma)$. This means that the homology class induced by $a_1x_1+\cdots+a_kx_k$ is the same as the homology class induced by $(a_1+\dots+a_k)x$. As the latter is in the image of $H_0(\pi)$, this homomorphism is an epimorphism. We can conclude that $H_0(\pi)$ is an isomorphism. Its inverse $H_0(p)$ is therefore an isomorphism too. We can now use Proposition 8.4.1 to conclude that $H_0(p, G)$ is also an isomorphism for any group G. $\square$

8.4.5 Corollary. $H_0(X, G) = \bigoplus_{\pi_0(X)} G$.

Proof. By Proposition 8.4.3, $H_0(X, G) = \bigoplus_{[x]\in\pi_0(X)} H_0(X_{[x]})$. Since $X_{[x]}$ is path connected, according to Proposition 8.4.4, $H_0(X_{[x]}) = G$. These two observations imply the corollary. $\square$

8.5 Homology and homotopy

Not only isomorphisms induce an isomorphism on homology groups. It turns out that the same is true for homotopy equivalences. This is one of the fundamental properties of the singular chain complexes and their homology groups.

8.5.1 Theorem. *Assume that maps $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ are homotopic. Then the homomorphisms of chain complexes $S_*(f) : S_*(X) \longrightarrow S_*(Y)$ and $S_*(g) : S_*(X) \longrightarrow S_*(Y)$ are also homotopic. In particular the induced homomorphisms $H_n(f, G) : H_n(X, G) \longrightarrow H_n(Y, G)$ and $H_n(g, G) : H_n(X, G) \longrightarrow H_n(Y, G)$ coincide for all n.*

Before we prove this theorem we are going to present some of its consequences.

8.5.2 Corollary. *(1) If $f : X \longrightarrow Y$ is a homotopy equivalence, then for any $n \geq 0$ and any group G, $H_n(f, G) : H_n(X, G) \longrightarrow H_n(Y, G)$ is an isomorphism.*

(2) If X and Y are homotopy equivalent spaces, then, for any $n \geq 0$ and any group G, the homology groups $H_n(X, G)$ and $H_n(Y, G)$ are isomorphic.

(3) If X is a contractible space, then:

$$H_n(X, G) = \begin{cases} G & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

Proof. Let $f : X \longrightarrow Y$ be a homotopy equivalence. Then there is $g : Y \longrightarrow X$ such that the compositions fg and gf are homotopic to identities. Thus according to Theorem 8.5.1 the homomorphisms $H_n(fg, G)$ and $H_n(gf, G)$ coincide with respectively $H_n(\mathrm{id}_Y, G)$ and $H_n(\mathrm{id}_X, G)$, and hence they are the identity homomorphisms. Since $H_n(fg, G) = H_n(f, G)H_n(g, G)$ and $H_n(gf, G) = H_n(g, G)H_n(f, G)$, it follows that $H_n(g, G)$ is the inverse of $H_n(f, G)$. The homomorphism $H_n(f, G)$ is therefore an isomorphism, and the first statement of the corollary is proven.

The second statement is a direct consequence of the first one.

The third statement follows from the second and Example 8.1.8. □

Combining the above corollary with Propositions 7.12.4 and 7.12.5 we get:

8.5.3 Corollary. *(1) If $D \subset \mathbf{R}^k$ is a convex non empty subspace (for example if $D = \Delta^{k-1}$ or D^k or $\mathbf{R}^k$), then:*

$$H_n(D, G) = \begin{cases} G & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

(2) Assume that $D \subset \mathbf{R}^k$ is convex and $x \in D$ a point for which there is $r > 0$ such that $B(x, r) \subset D$. Then, for any G and $n \geq 0$, the groups $H_n(D \setminus \{x\}, G)$ and $H_n(S^{k-1}, G)$ are isomorphic. In particular the groups $H_n(\mathbf{R}^k \setminus \{0\}, G)$ and $H_n(D^k \setminus \{0\}, G)$ are isomorphic to $H_n(S^{k-1}, G)$.

As a corollary of the homotopy invariance of homology we also get a statement which will be used to distinguish between $\mathbf{R}^k$'s for different k's.

8.5.4 Corollary. *If $\mathbf{R}^k$ is isomorphic to $\mathbf{R}^m$, then $H_n(S^{k-1})$ is isomorphic to $H_n(S^{m-1})$ for any n.*

Proof. Assume that $f : R^k \longrightarrow R^m$ is an isomorphism. Then the restriction $f : R^k \setminus \{0\} \longrightarrow R^m \setminus \{f(0)\}$ is also an isomorphism. Thus the induced homomorphism $H_n(f) : H_n(R^k \setminus \{0\}) \longrightarrow H_1(R^m \setminus \{f(0)\})$ is an isomorphism too. Since $R^k \setminus \{0\}$ and $R^m \setminus \{f(0)\}$ are homotopy equivalent to respectively S^{k-1} and S^{m-1}, the groups $H_n(S^{k-1})$ and $H_n(S^{m-1})$ are isomorphic for any n. □

Theorem 8.5.1 and Corollary 8.5.2 can be rephrased in terms of reduced homology:

8.5.5 Corollary. *(1) Assume that $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ are homotopic. Then the homomorphism $\tilde{H}_n(f, G) : \tilde{H}_n(X, G) \longrightarrow \tilde{H}_n(Y, G)$ and $\tilde{H}_n(g, G) : \tilde{H}_n(X, G) \longrightarrow \tilde{H}_n(Y, G)$ are the same for all n and all groups G.*

(2) If $f : X \longrightarrow Y$ is a homotopy equivalence, then $\tilde{H}_n(f, G) : \tilde{H}_n(X, G) \longrightarrow \tilde{H}_n(Y, G)$ is an isomorphism.

(3) If X and Y are homotopy equivalent, then, for any $n \geq 0$ and any groups G, the reduced homology groups $\tilde{H}_n(X, G)$ and $\tilde{H}_n(Y, G)$ are isomorphic.

(4) If X is contractible, then $\tilde{H}_n(X, G) = 0$ for any $n \geq 0$ and any G.

We now prove Theorem 8.5.1.

Proof of Theorem 8.5.1. We start by defying a sequence of homomorphisms $s_n : S_n(X) \longrightarrow S_{n+1}(Y)$. Let $S : X \times I \longrightarrow Y$ be a homotopy between f and g. Let $\sigma : \Delta^n \longrightarrow X$ be a singular simplex of dimension n in X. Consider the composition:

$$\Delta^{n+1} \xrightarrow{\tau_k} \Delta^n \times I \xrightarrow{\sigma \times \mathrm{id}} X \times I \xrightarrow{S} Y$$

where $\tau_k : \Delta^{n+1} \longrightarrow \Delta^n \times I$ is a linear simplex (see Example 8.1.3) given by:

$$\tau_k = \big((v_0, 0), \ldots, (v_k, 0), (v_k, 1), \ldots, (v_n, 1)\big)$$

(v_i denotes the i-th vertex of Δ^n). Define:

$$s_n(\sigma) := \Sigma_{k=0}^{n} (-1)^k S(\sigma \times \mathrm{id})(\tau_k)$$

We claim that the following equality holds:

$$S_n(g)(\sigma) - S_n(f)(\sigma) = \partial_{n+1} s_n(\sigma) + s_{n-1}\partial_n(\sigma)$$

The homomorphisms s_n, for $n \geq 0$, give therefore a homotopy between $S_*(f)$ and $S_*(g)$.

To show the above equality one uses the same argument for all n's. For clarity we will illustrate this calculation only in the case $n = 1$. The other cases are left as an exercise. Assume that $n = 1$. Note that:

$$\partial_2 s_1(\sigma) = \partial_2 S(\sigma \times \mathrm{id})(\tau_0) - \partial_2 S(\sigma \times \mathrm{id})(\tau_1) =$$

$$= S(\sigma \times \mathrm{id})\partial_2\big((v_0, 0), (v_0, 1), (v_1, 1)\big) - S(\sigma \times \mathrm{id})\partial_2\big((v_0, 0), (v_1, 0), (v_1, 1)\big) =$$

Note that:

$$S(\sigma \times \mathrm{id})\partial_2\big((v_0,0),(v_0,1),(v_1,1)\big) =$$

$$S(\sigma\times\mathrm{id})\big((v_0,1),(v_1,1)\big)-S(\sigma\times\mathrm{id})\big((v_0,0),(v_1,1)\big)+S(\sigma\times\mathrm{id})\big((v_0,0),(v_0,1)\big)$$

and

$$S(\sigma \times \mathrm{id})\partial_2\big((v_0,0),(v_1,0),(v_1,1)\big) =$$

$$S(\sigma\times\mathrm{id})\big((v_1,0),(v_1,1)\big)-S(\sigma\times\mathrm{id})\big((v_0,0),(v_1,1)\big)+S(\sigma\times\mathrm{id})\big((v_0,0),(v_1,0)\big)$$

It follows that

$$\partial_2 s_1(\sigma) = S(\sigma \times \mathrm{id})\big((v_0,1),(v_1,1)\big) - S(\sigma \times \mathrm{id})\big((v_0,0),(v_1,0)\big)+$$

$$+S(\sigma \times \mathrm{id})\big((v_0,0),(v_0,1)\big) - S(\sigma \times \mathrm{id})\big((v_1,0),(v_1,1)\big)$$

On the other hand:

$$s_0\partial_1(\sigma) = s_0(d_0\sigma) - s_0(d_1\sigma) =$$

$$= S(d_0\sigma \times \mathrm{id})\big((v_0,0),(v_0,1)\big) - S(d_1\sigma \times \mathrm{id})\big((v_0,0),(v_0,1)\big)$$

$$= S(\sigma \times \mathrm{id})\big((v_1,0),(v_1,1)\big) - S(\sigma \times \mathrm{id})\big((v_0,0),(v_0,1)\big)$$

It follows that:

$$\partial_2 s_1(\sigma) + s_0\partial_1(\sigma) = S(\sigma \times \mathrm{id})\big((v_0,1),(v_1,1)\big) - S(\sigma \times \mathrm{id})\big((v_0,0),(v_1,0)\big)$$

As S is a homotopy between f and g, we can then conclude that:

$$\partial_2 s_1(\sigma) + s_0\partial_1(\sigma) = S_1(g)(\sigma) - S_1(f)(\sigma)$$

□

8.6 Small singular simplices

So far we could only calculate homology in very few cases: the 0-th homology of an arbitrary space and all homology groups of contractible spaces. To calculate the homology of a sphere for example or a circle we need one more tool. This is the subject of this section. The chain complex $S_*(X)$ is simply too big for calculations. Our strategy is to find a much smaller subcomplex of $S_*(X)$ that has the same homology as $S_*(X)$.

Let X be a topological space. Let us fix an open cover of X, i.e., a family $\mathcal{U} = \{U_i\}$ of open subsets of X such that $X = \bigcup U_i$.

8.6.1 Definition. *An $\mathcal{U}$-singular simplex in X of dimension n is a continuous map $\sigma : \Delta^n \longrightarrow X$ such that for some i, the image $\sigma(\Delta^n)$ is included in U_i.*

The set of $\mathcal{U}$-singular simplexes of dimension n in X is denoted by $\Delta_n^{\mathcal{U}}(X)$. The symbol $S_n^{\mathcal{U}}(X)$ denotes the free abelian groups generated by the set $\Delta_n^{\mathcal{U}}(X)$.

Note that if $\sigma : \Delta^n \longrightarrow X$ is an $\mathcal{U}$-singular simplex in X, then so is $d_i\sigma$. It follows that the homomorphism $\partial_{n+1} : S_{n+1}(X) \longrightarrow S_n(X)$ takes elements of $S_{n+1}^{\mathcal{U}}(X)$ into $S_n^{\mathcal{U}}(X)$. In this way we obtain a chain subcomplex of $S_*(X)$ denoted by $S_*^{\mathcal{U}}(X)$. Explicitly it is given by:

$$\cdots \xrightarrow{\partial_{n+2}} S_{n+1}^{\mathcal{U}}(X) \xrightarrow{\partial_{n+1}} \cdots \xrightarrow{\partial_2} S_1^{\mathcal{U}}(X) \xrightarrow{\partial_1} S_0^{\mathcal{U}}(X)$$

It turns out that to calculate homology of a space we can use both the complex of singular simplices or the complex of $\mathcal{U}$-singular simplexes. This is the content of:

8.6.2 Theorem. *Let $\mathcal{U}$ be an open cover of a topological space X. Then the inclusion $i : S_*^{\mathcal{U}}(X) \subset S_*(X)$ is a homotopy equivalence, i.e., there is a chain complex homomorphism $s : S_*(X) \longrightarrow S_*^{\mathcal{U}}(X)$ such that the compositions $is : S_*(X) \longrightarrow S_*(X)$ and $si : S_*^{\mathcal{U}}(X) \longrightarrow S_*^{\mathcal{U}}(X)$ are homotopic to identities.*

This theorem together with the homotopy invariance of homology (Theorem 8.5.1) will be our key tools to calculate homology of spaces. The rest of these notes explain how to use this theorem in practise. We start with:

8.6.3 Corollary. *For any group G and any $n \geq 0$, the groups $H_n(X, G)$ and $H_n(S_*^{\mathcal{U}}(X) \otimes G)$ are isomorphic.*

8.6.4 Excercise. Consider the following subsets of S^1:

$$U_0 := \{(x, y) \in S^1 \mid y > -1/2\} \qquad U_1 := \{(x, y) \in S^1 \mid y < 1/2\}$$

They form an open cover $\mathcal{U} = \{U_0, U_1\}$ of S^1. Consider also a singular simplex $\sigma : I \longrightarrow S^1$ given by the formula $\sigma(t) = (\sin(2t\pi), \cos(2t\pi))$. Is σ a $\mathcal{U}$-singular simplex? Find a singular simplex $\tau \in S_1^{\mathcal{U}}(S^1)$ such that $\tau - \sigma$ is a boundary in $S_1(S^1)$, i.e., it is in the image of $\partial_2 : S_2(S^1) \longrightarrow S_1(S^1)$.

8.7 Reduced small singular simplices

Let $\mathcal{U}$ be an open cover of X. Define $\tilde{S}_*^{\mathcal{U}}(X)$ to be the kernel of the composition:

$$S_*^{\mathcal{U}}(X) \subset S_*(X) \xrightarrow{S_*(p)} S_*(D^0)$$

The group $\tilde{S}^{\mathcal{U}}_n(X)$ is then the subgroup of $S_*(X)$ consisting of these combinations of $\mathcal{U}$-singular simplices $a_1\sigma_1+\cdots+a_k\sigma_k$ for which the sum $a_1+\cdots+a_k = 0$. It is a subgroup of the reduced complex $\tilde{S}_n(X)$.

8.7.1 Excercise. Assume that X is not an empty space. Show that the chain complexes $S_*(X)$, $\tilde{S}_*(X)$, $S^{\mathcal{U}}_*(X)$, $\tilde{S}^{\mathcal{U}}_*(X)$, and $S_*(D^0)$ fit into the following commutative diagram of chain complexes whose rows are split exact:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{S}^{\mathcal{U}}_*(X) \arrow[r, hook] \arrow[d, hook] & S^{\mathcal{U}}_*(X) \arrow[r] \arrow[d, hook] & S_*(D^0) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \tilde{S}_*(X) \arrow[r, hook] & S_*(X) \arrow[r, "S_*(p)"] & S_*(D^0) \arrow[r] & 0
\end{tikzcd}$$

8.7.2 Excercise. Show that the inclusion $\tilde{S}^{\mathcal{U}}_*(X) \subset \tilde{S}_*(X)$ is a homotopy equivalence of chain complexes (use Theorem 8.6.2).

Using the above exercises and Theorem 8.6.2, we can rephrase Proposition 8.3.3 in terms of the $\mathcal{U}$-singular simplices:

8.7.3 Proposition. *Let X be a non empty space, $\mathcal{U}$ its open cover, and G a group.*

(1) $H_0(X,G)$ is isomorphic to $G \oplus H_0(\tilde{S}^{\mathcal{U}}_*(X) \otimes G)$.

(2) For $n > 0$, the homomorphism $H_n(\tilde{S}^{\mathcal{U}}_*(X) \otimes G) \longrightarrow H_n(X,G)$, induced by the inclusion $\tilde{S}^{\mathcal{U}}_*(X) \subset S_*(X)$, is an isomorphism.

The key point of the above proposition is that to calculate homology of X, instead of using the complex $S_*(X)$, me can use $\tilde{S}^{\mathcal{U}}_*(X)$ for some open cover $\mathcal{U}$ of X. Why is it easier to use the complexes $\tilde{S}^{\mathcal{U}}_*(X)$ or $S^{\mathcal{U}}_*(X)$? The answer is that they often lead to exact sequences. Let us illustrate this in the basic case of an open cover $\mathcal{U} = \{U_0, U_1\}$ of X that consists of only two open subsets. This will be the case used in the rest of these notes. Let $V = U_0 \cap U_1$ and assume that V is not empty. We are going to use the following symbols to denote the appropriate inclusions:

$$\begin{tikzcd}
V \arrow[r, hook, "j_0"] \arrow[d, hook, "j_1"'] & U_0 \arrow[d, hook, "i_1"] \\
U_1 \arrow[r, hook, "i_0"] & X
\end{tikzcd}$$

Observe that the inclusions $i_0 : U_0 \subset X$ and $i_1 : U_1 \subset X$ map singular simplices in U_i into $\mathcal{U}$-singular simplexes of X. Moreover any $\mathcal{U}$-singular simplexes of X is in the image of $S_*(i_0)$ or $S_*(i_1)$. This means that the following homomorphism is an epimorphism:

$$S_*(i_0) \oplus S_*(i_1) : S_*(U_0) \oplus S_*(U_1) \longrightarrow S_*^{\mathcal{U}}(X)$$

What is the kernel of this homomorphism? A pair $(x, y) \in S_*(U_0) \oplus S_*(U_1)$ is in the kernel of $S_*(i_0) \oplus S_*(i_1)$ if and only if $S_*(i_0)(x) = -S_*(i_1)(y)$. This can happen if and only if x is in the image of both $S_*(j_0) : S_*(V) \longrightarrow S_*(U_0)$ and $S_*(j_1) : S_*(V) \longrightarrow S_*(U_1)$. This leads to an exact sequence of chain complexes:

$$0 \longrightarrow S_*(V) \xrightarrow{(S_*(j_0), -S_*(j_1))} S_*(U_0) \oplus S_*(U_1) \xrightarrow{S_*(i_0)+S_*(i_1)} S_*^{\mathcal{U}}(X) \longrightarrow 0$$

8.7.4 Excercise. Show that the above sequence is a part of the following larger commutative diagram of chain complexes whose all rows and columns are exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tilde{S}_*(V) \arrow[r, "(\tilde{S}_*(j_0), -\tilde{S}_*(j_1))"] \arrow[d, hook] & \tilde{S}_*(U_0) \oplus \tilde{S}_*(U_1) \arrow[r, "\tilde{S}_*(i_0)+\tilde{S}_*(i_1)"] \arrow[d, hook] & \tilde{S}_*^{\mathcal{U}}(X) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & S_*(V) \arrow[r, "(S_*(j_0), -S_*(j_1))"] \arrow[d, "S_*(p)"] & S_*(U_0) \oplus S_*(U_1) \arrow[r, "S_*(i_0)+S_*(i_1)"] \arrow[d, "S_*(p)\oplus S_*(p)"] & S_*^{\mathcal{U}}(X) \arrow[r] \arrow[d, "S_*(p)"] & 0 \\
0 \arrow[r] & S_*(D^0) \arrow[r, "(\mathrm{id}, -\mathrm{id})"] \arrow[d] & S_*(D^0) \oplus S_*(D^0) \arrow[r, "\mathrm{id}+\mathrm{id}"] \arrow[d] & S_*(D^0) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since all the groups in the above diagram are free, we can tensor this diagram with a group G and get analogous diagram whose rows and columns are also exact. We will be interested only in its top row. For any group G we then have an exact sequence of chain complexes:

$$0 \longrightarrow \tilde{S}_*(V) \otimes G \longrightarrow (\tilde{S}_*(U_0) \otimes G) \oplus (\tilde{S}_*(U_1) \otimes G) \longrightarrow \tilde{S}_*^{\mathcal{U}}(X) \otimes G \longrightarrow 0$$

Such an exact sequence of chain complexes leads to a long exact sequence of their homologies:

$$\begin{tikzcd}
 & & \cdots \arrow[r] & \tilde{H}_2(X,G) \arrow[dlll] \\
\tilde{H}_1(V,G) \arrow[rr, "{(\tilde{H}_1(j_0,G),-\tilde{H}_0(j_1,G))}"'] & & \tilde{H}_1(U_0,G)\oplus\tilde{H}_1(U_1,G) \arrow[r, "\tilde{H}_1(i_0,G)+\tilde{H}_1(i_1,G)"] & \tilde{H}_1(X,G) \arrow[dlll] \\
\tilde{H}_0(V,G) \arrow[rr, "{(\tilde{H}_0(j_0,G),-\tilde{H}_0(j_1,G))}"'] & & \tilde{H}_0(U_0,G)\oplus\tilde{H}_0(U_1,G) \arrow[r, "\tilde{H}_0(i_0,G)+\tilde{H}_0(i_1,G)"] & \tilde{H}_0(X,G) \arrow[dlll] \\
0 & & &
\end{tikzcd}$$

We can now use this exact sequence to relate homologies of X, U_0, U_1, and V. For example, assume that U_0 and U_1 are contractible spaces. In this case $\tilde{H}_n(U_0,G) = \tilde{H}_n(U_1,G) = 0$ for any n and thus the above exact sequence gives:

8.7.5 Proposition. *Let $\mathcal{U} = \{U_0, U_1\}$ be an open cover of X and G be a group. Assume that $V = U_0 \cap U_1$ is not empty and that U_0 and U_1 are contractible. Then, $\tilde{H}_0(X,G) = 0$ and, for $n > 0$, $\tilde{H}_n(X,G) = \tilde{H}_{n-1}(V,G)$.*

If only one of $\{U_0, U_1\}$ is contractible, for example U_1, then again since $\tilde{H}_n(U_1,G) = 0$, for all n, the above exact sequence gives:

8.7.6 Proposition. *Let $\mathcal{U} = \{U_0, U_1\}$ be an open cover of X and G be a group. Assume that $V = U_0 \cap U_1$ is not empty and that U_1 is contractible. Then there is an exact sequence:*

$$\begin{tikzcd}
 & & \cdots \arrow[r] & \tilde{H}_2(X,G) \arrow[dlll] \\
\tilde{H}_1(V,G) \arrow[rr, "\tilde{H}_1(j_0,G"'] & & \tilde{H}_1(U_0,G) \arrow[r, "\tilde{H}_1(i_0,G)"] & \tilde{H}_1(X,G) \arrow[dlll] \\
\tilde{H}_0(V,G) \arrow[rr, "\tilde{H}_0(j_0,G)"'] & & \tilde{H}_0(U_0,G) \arrow[r, "\tilde{H}_0(i_0,G)"] & \tilde{H}_0(X,G) \arrow[dlll] \\
0 & & &
\end{tikzcd}$$

8.8 Homology of spheres

We already know that, since S^0 is a discreet space consisting of two points, then:

$$H_n(S^0, G) = \begin{cases} G \oplus G & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

$$\tilde{H}_n(S^0, G) = \begin{cases} G & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

The aim of this section is to prove:

8.8.1 Theorem. *Let G be a group. Then:*

$$\tilde{H}_n(S^k, G) = \begin{cases} G & \textit{if } n = k \\ 0 & \textit{if } n \neq k \end{cases}$$

Proof. The proof is by induction on k. The case $k = 0$ has already been proven. Let $k > 0$. We are going to use the following open subsets of S^k:

$$U_0 = \{(x_1, \ldots x_{k+1}) \in S^k \mid x_{k+1} > -1/2\}$$

$$U_1 = \{(x_1, \ldots x_{k+1}) \in S^k \mid x_{k+1} < 1/2\}$$

$$V = U_0 \cap U_1 = \{(x_1, \ldots x_{k+1}) \in S^k \mid -1/2 < x_{k+1} < 1/2\}$$

First we need to understand the homotopy type of these subspaces:

8.8.2 Excercise. (1) Show that U_0 and U_1 are contractible spaces.

(2) Show that U_0 and U_1 are isomorphic to $\mathbf{R}^k$.

(3) Show that the following inclusion is a homotopy equivalence:

$$S^{k-1} \ni (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0) \in V$$

From the above exercise it follows that the cover $\mathcal{U} = \{U_0, U_1\}$ consists of contractible spaces. We can then use Proposition 8.7.5 to conclude that $\tilde{H}_0(S^k, G) = 0$ and, for $n > 0$, $\tilde{H}_n(S^k, G) = \tilde{H}_{n-1}(V, G)$. However since V is homotopic to S^{k-1}, we get that for $n > 0$:

$$\tilde{H}_n(S^k, G) = \tilde{H}_{n-1}(S^{k-1}, G)$$

We can now use the inductive assumption to finish the proof of the theorem. □

8.8.3 Corollary. *Let $k > 0$ and G be a group. Then:*

$$H_n(S^k, G) = \begin{cases} G & \text{if } n = 0 \text{ or } n = k \\ 0 & \text{if } n \neq 0 \text{ and } n \neq k \end{cases}$$

8.8.4 Excercise. Let $\sigma : I \longrightarrow S^1$ be a map given by the formula $\sigma(t) = \big(\sin(2t\pi), \cos(2t\pi)\big)$. It is a singular simplex in $S_1(S^1)$. Show that it is a cycle, i.e., it is in the kernel of $\partial_1 : S_1(S^1) \longrightarrow S_0(S^1)$. Show moreover, that its homology class $[\sigma] \in H_1(S^1) = \mathbf{Z}$ is a generator.

8.8.5 Excercise. Let $\tau : I \longrightarrow S^1$ be a map given by the formula $\sigma(t) = \big(\sin(4t\pi), \cos(4t\pi)\big)$. It is a singular simplex in $S_1(S^1)$. Show that it is a cycle, i.e., it is in the kernel of $\partial_1 : S_1(S^1) \longrightarrow S_0(S^1)$. Show moreover that $\tau - 2\sigma$, were σ is defined as in Exercise 8.8.4, is a boundary, i.e., it is in the image of $\partial_2 : S_2(S^1) \longrightarrow S_1(S^1)$. Conclude that the homology class $[\tau] \in H_1(S^1)$ is equal to $2[\sigma] \in H_1(S^1)$.

8.8.6 Excercise. Let us think about S^1 as a subspace of complex numbers $\mathbf{C}$, of length 1. Let $\alpha : S^1 \longrightarrow S^1$ be the squaring map $\alpha(z) := z^2$. Show that $H_1(\alpha) : H_1(S^1) \longrightarrow H_1(S^1)$ is multiplication by 2. Show that the same holds for the homomorphism $H_1(\alpha, G) : H_1(S^1, G) \longrightarrow H_1(S^1, G)$, for any group G.

8.8.7 Excercise. Let σ be as defined in Exercise 8.8.4. Let $\sigma' : I \longrightarrow S^1$ be given by $\sigma'(t) = \big(\sin(2\pi(1-t), \cos(2\pi(1-t)\big)$. Show that σ' is a cycle. Show further that $\sigma + \sigma'$ is a boundary in $S_1(S^1)$, i.e., it is in the image of $\partial_2 : S_2(S^1) \longrightarrow S_1(S^1)$. Conclude that the homology class $[\sigma'] \in H_1(S^1)$ is a generator and $[\sigma'] = -[\sigma]$.

8.8.8 Excercise. Let $\beta : S^1 \longrightarrow S^1$ be the map given by the formula $\beta(x_1, x_2) = (x_1, -x_2)$. Show that $H_1(\beta) : H_1(S^1) \longrightarrow H_1(S^1)$ is the multiplication by -1.

8.9 Some geometric applications

In this section we present some geometric applications and corollaries of the calculation presented in Theorem 8.8.1. We start with:

8.9.1 Proposition. *S^k is homotopy equivalent to S^m if and only if $k = m$.*

Proof. If $k = m$, then S^k is even isomorphic to S^m. Assume that S^k is homotopy equivalent to S^m. Then the homology groups $\tilde{H}_n(S^k)$ and $\tilde{H}_n(S^m)$ are then isomorphic for all n. According to Theorem 8.8.1, this can happen if and only if $k = m$. $\square$

We have now enough tools to show:

8.9.2 Proposition. $\mathbf{R}^k$ *is isomorphic to* $\mathbf{R}^m$ *if and only if* $k = m$.

Proof. If $k = m$, then obviously $\mathbf{R}^k$ is isomorphic to $\mathbf{R}^m$. Assume that $\mathbf{R}^k$ and $\mathbf{R}^m$ are isomorphic. By Corollary 8.5.4 we then have that $H_n(S^{k-1})$ and $H_n(S^{m-1})$ are isomorphic groups for all n. Using Proposition 8.9.1 we can then conclude that $k - 1 = m - 1$, and hence $k = m$. □

8.9.3 Excercise. Let M be a manifold of dimension m and N be a manifold of dimension k (see Definition 7.11.1). Show that if M and N are isomorphic, then $m = k$.

8.9.4 Proposition. *Let* $k \geq 1$. *There is no map* $r : D^k \longrightarrow S^{k-1}$ *for which the composition of the inclusion* $S^{k-1} \subset D^k$ *and* r *is homotopic to* $id : S^{k-1} \longrightarrow S^{k-1}$.

Proof. Assume that such a map exists. Consider its effect on homology:

$$\tilde{H}_{k-1}(S^{k-1}) \longrightarrow \tilde{H}_{k-1}(D^k) \xrightarrow{\tilde{H}_{k-1}(r)} \tilde{H}_{k-1}(S^{k-1})$$

Note that this composition must be the identity. Thus we would get a commutative diagram of the form:

$$\begin{tikzcd} \mathbf{Z} \arrow[r] \arrow[rr, bend right, "\mathrm{id}"] & 0 \arrow[r] & \mathbf{Z} \end{tikzcd}$$

This however is impossible (why?). □

8.9.5 Proposition. *Let* $f : D^k \longrightarrow D^k$ *be a continuous map. Then there is a point* $x \in D^k$, *such that* $f(x) = x$.

Proof. The case $k = 0$ is obvious. Assume that $k \geq 1$ and that there is no such point, i.e., for all $x \in D^k$, $f(x) \neq x$. We can then define a map $r : D^k \longrightarrow S^{k-1}$ by the formula:

$$r(x) := \frac{x - f(x)}{|x - f(x)|}$$

We are going to show that the composition of the inclusion $S^{k-1} \subset D^k$ and r is homotopic to the identity. We can then use Proposition 8.9.4 to get a contradiction. Define the homotopy $H : S^{k-1} \times I \longrightarrow S^{k-1}$ by the formula:

$$H(x, t) := \frac{x - tf(x)}{|x - tf(x)|}$$

This is well defined since $x - tf(x) \neq 0$ (why?). Note that $H(x, 0) = x$ and $H(x, 1)$ is the discussed composition.

8.9.6 Excercise. Let $f : S^{2k} \longrightarrow S^{2k}$ be a continuous map. Show that there is a point $x \in S^{2k}$ such that either $f(x) = x$ or $f(x) = -x$. Is the same true for an odd dimensional sphere?

$\square$

8.10 The homology of the antipodal map

The aim of this section is to determined the effect on the k-th homology group of the antipodal map $\mathrm{an} : S^k \longrightarrow S^k$. By definition $\mathrm{an} : S^k \longrightarrow S^k$ is the map given by the formula:

$$\mathrm{an}(x_1, \ldots, x_{k+1}) := (-x_1, \ldots, -x_{k+1})$$

Let $k \geq 1$ and $k+1 \geq i \geq 1$. Define the map $f_i : S^k \longrightarrow S^k$ to be given by the formula:

$$f_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{k+1}) = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_{k+1})$$

We first determine the effect of f_i on the k-homology group:

8.10.1 Proposition. *Let $k \geq 1$ and $k+1 \geq i \geq 1$. For any group G, the homomorphism $H_k(f_i, G) : H_k(S^k, G) \longrightarrow H_k(S^k, G)$ is the multiplication by -1.*

Proof. The proof is by induction on k. The case $k = 1$ has been the subject of Exercise 8.8.8. Assume that $k > 1$. Let us choose $j \neq i$. As in the proof of Theorem 8.8.1, let:

$$U_0 = \{(x_1, \ldots x_{k+1}) \in S^k \mid x_j > -1/2\}$$
$$U_1 = \{(x_1, \ldots x_{k+1}) \in S^k \mid x_j < 1/2\}$$
$$V = U_0 \cap U_1 = \{(x_1, \ldots x_{k+1}) \in S^k \mid -1/2 < x_j < 1/2\}$$

Note that $f_i : S^k \longrightarrow S^k$ maps U_0 to U_0 and U_1 to U_1. Moreover restricted to $S^{k-1} = \{(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_{k+1}) \in V \subset S^k\}$ the map f_i is the analogous map to f_i, but for S^{k-1}. Thus by the inductive assumption $H_{k-1}(f_i, G) : H_{k-1}(V, G) \longrightarrow H_{k-1}(V, G)$ is multiplication by -1. Consider next the following commutative diagram of chain complexes, whose rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & S_*(V) \arrow[r, "{(S_*(j_0), -S_*(j_1))}"] \arrow[d, "S_*(f_i)"] & S_*(U_0) \oplus S_*(U_1) \arrow[r, "S_*(i_0) \oplus S_*(i_1)"] \arrow[d, "S_*(f_i) \oplus S_*(f_i)"] & S_*^{\mathcal{U}}(S^k) \arrow[r] \arrow[d, "S_*(f_i)"] & 0 \\
0 \arrow[r] & S_*(V) \arrow[r, "{(S_*(j_0), -S_*(j_1))}"] & S_*(U_0) \oplus S_*(U_1) \arrow[r, "S_*(i_0) \oplus S_*(i_1)"] & S_*^{\mathcal{U}}(S^k) \arrow[r] & 0
\end{tikzcd}$$

This leads to a commutative square, as in Proposition 8.7.5, with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H_k(S^k,G) \arrow[r, "\partial"] \arrow[d, "H_k(f_i,G)"] & H_{k-1}(V,G) \arrow[r] \arrow[d, "H_{k-1}(f_i,G)"] & 0 \\
0 \arrow[r] & H_k(S^k,G) \arrow[r, "\partial"] & H_{k-1}(V,G) \arrow[r] & 0
\end{tikzcd}$$

where $\partial : H_k(S^k, G) \longrightarrow H_{k-1}(V, G)$ is the connecting homomorphism induces by the short exact sequence of chain complexes above. Since by the inductive assumption $H_{k-1}(f_i, G) : H_{k-1}(V, G) \longrightarrow H_{k-1}(V, G)$ is multiplication by -1 and ∂ is an isomorphism, then $H_k(f_i, G) : H_k(S^k, G) \longrightarrow H_k(S^k, G)$ is also the multiplication by -1 homomorphism. $\square$

8.10.2 Corollary. *Let $k \geq 1$. The homomorphism $H_k(an, G) : H_k(S^k, G) \longrightarrow H_k(S^k, G)$ is multiplication by $(-1)^{k+1}$.*

Proof. Note that an is the composition of f_i for $k + 1 \geq i \geq 1$, i.e.,:

$$\mathrm{an} = f_{k+1} f_k \cdots f_1$$

Thus:

$$H_k(\mathrm{an}, G) = H_k(f_{k+1}, G) H_k(f_k, G) \cdots H_k(f_1, G)$$

Since, by Proposition 8.10.1, $H_k(f_i, G) : H_k(S^k, G) \longrightarrow H_k(S^k, G)$ is the multiplication by -1, for all i, the corollary follows. $\square$

8.11 The effect on homology of a cell attachment

Assume that we know the homology of X and the effect on homology groups of the map $\alpha : S^{k-1} \longrightarrow X$. Is this information enough to determine the homology of the space $X \cup_\alpha D^k$ (see Section 7.9)? Let $i : X \subset X \cup_\alpha D^k$ be the inclusion map.

8.11.1 Theorem. *Let $k \geq 1$, $\alpha : S^{k-1} \longrightarrow X$ be a map, and G be a group.*

(1) If $n \neq k - 1$ and $n \neq k$, then $H_n(i, G) : H_n(X, G) \longrightarrow H_n(X \cup_\alpha D^k, G)$ is an isomorphism.

(2) There is an exact sequence of abelian groups:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{H}_k(X,G) \arrow[r, "\tilde{H}_k(i,G)"] & \tilde{H}_k(X \cup_\alpha D^k, G) \arrow[dll] \\
\tilde{H}_{k-1}(S^{k-1},G) \arrow[r, "\tilde{H}_{k-1}(\alpha,G)"'] & \tilde{H}_{k-1}(X,G) \arrow[r, "\tilde{H}_{k-1}(i,G)"] & \tilde{H}_{k-1}(X \cup_\alpha D^k, G) \arrow[dll] \\
0 & &
\end{tikzcd}$$

Proof. Define the following open subsets of $X \cup_\alpha D^k$:

$$U_0 := X \coprod \{x \in B^k \mid |x| > 1/3\}$$

$$U_1 := \{x \in B^k \mid 2/3 > |x|\}$$

$$V := U_0 \cap U_1 = \{x \in B^k \mid 2/3 > |x| > 1/3\}$$

We need to understand the homotopy type of these subspaces:

8.11.2 Excercise. Show that the inclusion $X \subset U_0$ is a homotopy equivalence.

8.11.3 Excercise. Show that U_1 is a contractible space and that the inclusion:

$$\{x \in V \mid |x| = 1/2\} \subset V$$

is a homotopy equivalence. Conclude that V is homotopy equivalent to S^{k-1}.

8.11.4 Excercise. Let $S := \{x \in V \mid |x| = 1/2\}$ and $f : S^{k-1} \longrightarrow S$ be defined by the formula $f(x) = x/2$. Show that the following compositions are homotopic:

$$S^{k-1} \xrightarrow{f} S \subset V \subset U_0$$

$$S^{k-1} \xrightarrow{\alpha} X \subset U_0$$

It is clear that $\mathcal{U} := \{U_0, U_1\}$ is an open cover of $X \cup_\alpha D^k$. Since U_1 is contractible and V is not empty we can use Proposition 8.7.6 to get an exact seduce:

$$\begin{tikzcd}
 & \cdots \arrow[r] & \tilde{H}_2(X \cup_\alpha D^k, G) \arrow[dll] \\
\tilde{H}_1(V,G) \arrow[r, "\tilde{H}_1(j_0,G)"'] & \tilde{H}_1(U_0,G) \arrow[r, "\tilde{H}_1(i_0,G)"] & \tilde{H}_1(X \cup_\alpha D^k, G) \arrow[dll] \\
\tilde{H}_0(V,G) \arrow[r, "\tilde{H}_0(j_0,G)"'] & \tilde{H}_0(U_0,G) \arrow[r, "\tilde{H}_0(i_0,G)"] & \tilde{H}_0(X \cup_\alpha D^k, G) \arrow[dll] \\
0 & &
\end{tikzcd}$$

Since V is homotopy equivalent to S^{k-1}, if $n \neq k$ and $n \neq k-1$, the homology groups $\tilde{H}_n(V,G)$ and $\tilde{H}_{n-1}(V,G)$ are trivial. It follows that the homomorphism $\tilde{H}_n(i_0,G) : \tilde{H}_n(U_0,G) \longrightarrow \tilde{H}_n(X \cup_\alpha D^k, G)$ is an isomorphism. As the inclusion $X \subset U_0$ is a homotopy equivalence, we can conclude that, for $n \neq k$ and $n \neq k-1$, the homomorphism $\tilde{H}_n(i,G) : \tilde{H}_n(X,G) \longrightarrow \tilde{H}_n(X \cup_\alpha D^k, G)$ is an isomorphism. It then follows that so is $H_n(i,G) : H_n(X,G) \longrightarrow H_n(X \cup_\alpha D^k, G)$.

In the case $n = k$ or $n = k-1$, the above exact sequence becomes:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{H}_k(U_0,G) \arrow[r, "\tilde{H}_k(i_0,G)"] & \tilde{H}_k(X \cup_\alpha D^k, G) \arrow[dll] \\
\tilde{H}_{k-1}(V,G) \arrow[r, "\tilde{H}_{k-1}(j_0,G)"'] & \tilde{H}_{k-1}(U_0,G) \arrow[r, "\tilde{H}_{k-1}(i_0,G)"] & \tilde{H}_{k-1}(X \cup_\alpha D^k, G) \arrow[dll] \\
0 & &
\end{tikzcd}$$

Using Exercise 8.11.4, we get the claimed exact sequence. $\square$

8.12 The integral homology groups of the projective spaces

In this section we will determine the integral homology of the projective spaces $\mathbf{RP}^n$. Since $\mathbf{RP}^1 = S^1$, we then have:

$$\tilde{H}_n(\mathbf{RP}^1, \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

The aim of this section is to prove:

8.12.1 Theorem. *Let $k \geq 0$. Then:*

(1) if k is even

$$\tilde{H}_n(\mathbf{RP}^k, \mathbf{Z}) = \begin{cases} \mathbf{Z}/2 & \textit{if } n \textit{ is odd and } n < k \\ 0 & \textit{if } n \textit{ is even or } n \geq k \end{cases}$$

(2) if k is odd

$$\tilde{H}_n(\mathbf{RP}^k, \mathbf{Z}) = \begin{cases} \mathbf{Z}/2 & \textit{if } n \textit{ is odd and } n < k \\ 0 & \textit{if } n \textit{ is even or } n > k \\ \mathbf{Z} & \textit{if } n = k \end{cases}$$

First, we are going to prove several supporting statements. Let $k \geq 1$. Recall that $\pi : S^{k-1} \longrightarrow \mathbf{RP}^{k-1}$ is the map that sends the vector v to the line in $\mathbf{R}^k$ spanned by v. This map is used to build the next projective space. According to Proposition 7.10.3 there is an isomorphism:

$$\mathbf{RP}^k = \mathbf{RP}^{k-1} \cup_\pi D^k$$

Let $i : \mathbf{RP}^{k-1} \subset \mathbf{RP}^k$ be the standard inclusion. According to Theorem 8.11.1.(1) we get:

8.12.2 Lemma. *Let $k \geq 1$. If $n \neq k-1$ and $n \neq k$, then $\tilde{H}_n(i) : \tilde{H}_n(\mathbf{RP}^{k-1}) \longrightarrow \tilde{H}_n(\mathbf{RP}^k)$ is an isomorphism. In particular, for $n \geq 2$:*

$$0 = \tilde{H}_n(S^1) = \tilde{H}_n(\mathbf{RP}^1) = \cdots = \tilde{H}_n(\mathbf{RP}^{n-1})$$

Theorem 8.11.1.(2) gives also an exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{H}_k(\mathbf{RP}^{k-1}) \arrow[r, "\tilde{H}_k(i)"] & \tilde{H}_k(\mathbf{RP}^k) \arrow[dll, "\partial"] \\
\tilde{H}_{k-1}(S^{k-1}) \arrow[r, "\tilde{H}_{k-1}(\pi)"'] & \tilde{H}_{k-1}(\mathbf{RP}^{k-1}) \arrow[r, "\tilde{H}_{k-1}(i)"] & \tilde{H}_{k-1}(\mathbf{RP}^k) \arrow[dll] \\
0 & &
\end{tikzcd}$$

With a help of Lemma 8.12.2 this sequence becomes:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \tilde{H}_k(\mathbf{RP}^k) \arrow[dll, "\partial"] \\
\tilde{H}_{k-1}(S^{k-1}) \arrow[r, "\tilde{H}_{k-1}(\pi)"'] & \tilde{H}_{k-1}(\mathbf{RP}^{k-1}) \arrow[r, "\tilde{H}_{k-1}(i)"] & \tilde{H}_{k-1}(\mathbf{RP}^k) \arrow[dll] \\
0 & &
\end{tikzcd}$$

It follows that $\tilde{H}_k(\mathbf{RP}^k)$ is a subgroup of $\tilde{H}_k(S^{k-1}) = \mathbf{Z}$, and consequently it is either 0 or is isomorphic to $\mathbf{Z}$. We are going to show that this depends on the parity of k.

To calculate the homology of projective spaces we thus need to understand the homomorphisms $\tilde{H}_k(\pi) : \tilde{H}_k(S^k) \longrightarrow \tilde{H}_k(\mathbf{RP}^k)$ and $\partial : \tilde{H}_k(\mathbf{RP}^k) \longrightarrow \tilde{H}_{k-1}(S^{k-1})$ for various k's. To do that we will analyze the following open covers of $\mathbf{RP}^k$ and S^k. Let $k \geq 1$. Let $\mathcal{U} = \{U_0, U_1\}$ be an open cover of $\mathbf{RP}^k$ given, as in the previous section, by:

$$U_0 := \mathbf{RP}^{k-1} \coprod \{x \in B^k \mid |x| > 1/3\}$$

$$U_1 := \{x \in B^k \mid 2/3 > |x|\}$$

Define:

$$\begin{array}{rcl}
V & := & U_0 \cap U_1 = \{x \in B^k \mid 2/3 > |x| > 1/3\} \\
S & := & \{x \in V \mid |x| = 1/\sqrt{3}\} \subset V \\
W_0 & := & \pi^{-1}(U_0) \subset S^k \\
W_1 & := & \pi^{-1}(U_1) \subset S^k \\
Z_0 & := & \{(x_1, \ldots, x_{k+1}) \in S^k \mid x_{k+1} > \sqrt{5}/3\} \subset S^k \\
Z_1 & := & \{(x_1, \ldots, x_{k+1}) \in S^k \mid x_{k+1} < -\sqrt{5}/3\} \subset S^k \\
Y & := & W_0 \cap W_1 \subset S^k \\
Y_0 & := & \{(x_1, \ldots, x_{k+1}) \in S^k \mid \sqrt{8}/3 > x_{k+1} > \sqrt{5}/3\} \subset S^k \\
Y_1 & := & \{(x_1, \ldots, x_{k+1}) \in S^k \mid -\sqrt{8}/3 < x_{k+1} < -\sqrt{5}/3\} \subset S^k \\
S_0 & := & \{(x_1, \ldots, x_{k+1}) \in S^k \mid x_{k+1} = \sqrt{6}/3\} \subset S^k \\
S_1 & := & \{(x_1, \ldots, x_{k+1}) \in S^k \mid x_{k+1} = -\sqrt{6}/3\} \subset S^k
\end{array}$$

8.12.3 Excercise. Show that:

(1) $W_0 = \{(x_1, \ldots, x_{k+1}) \in S^k \mid |x_{k+1}| < \sqrt{8}/3\}$

(2) $W_1 = Z_0 \cup Z_1$

(3) $Y = Y_0 \cup Y_1$

8.12.4 Excercise. (1) Show that $S \subset V$ and all the following inclusions are homotopy equivalences:

$$S_0 \hookrightarrow Y_0 \hookrightarrow W_0 \hookleftarrow Y_1 \hookleftarrow S_1$$

(2) Z_0 and Z_1 are contractible spaces.

8.12.5 Excercise. (1) Show that π takes S_0 and S_1 into S and that the induced maps $\pi : S_0 \longrightarrow S$ and $\pi : S_1 \longrightarrow S$ are isomorphisms.

(2) Show that the following formulas define continuous maps:

$$\begin{array}{ccc}
s_0 : S^{k-1} \longrightarrow S_0 & s_1 : S^{k-1} \longrightarrow S_1 & s : S^{k-1} \longrightarrow S \\
s_0(x) := (x/\sqrt{3}, \sqrt{6}/3) & s_1(x) := (x/\sqrt{3}, -\sqrt{6}/3) & s(x) = x/\sqrt{3}
\end{array}$$

(3) Prove that the maps s_0, s_1, and s are isomorphisms and show that the following diagrams commute:

$$\begin{tikzcd} S^{k-1} \arrow[r, "s_0"] \arrow[d, "\mathrm{id}"'] & S_0 \arrow[d, "\pi"] \\ S^{k-1} \arrow[r, "s"] & S \end{tikzcd} \qquad \begin{tikzcd} S^{k-1} \arrow[r, "s_1"] \arrow[d, "\mathrm{an}"'] & S_1 \arrow[d, "\pi"] \\ S^{k-1} \arrow[r, "s"] & S \end{tikzcd}$$

where $\mathrm{an} : S^{k-1} \longrightarrow S^{k-1}$ is the antipodal map.

8.12.6 Excercise. Show that there is a commutative diagram:

$$\begin{tikzcd} \tilde{H}_{k-1}(S^{k-1}) \oplus \tilde{H}_{k-1}(S^{k-1}) \arrow[r, "\tilde{H}_{k-1}(s_0)\oplus\tilde{H}_{k-1}(s_1)"] \arrow[d, "1+(-1)^k"] & \tilde{H}_{k-1}(S_0) \oplus \tilde{H}_{k-1}(S_1) \arrow[r] \arrow[d, "\tilde{H}_{k-1}(\pi)+\tilde{H}_{k-1}(\pi)"'] & \tilde{H}_{k-1}(Y) \arrow[d, "\tilde{H}_{k-1}(\pi)"'] \\ \tilde{H}_{k-1}(S^{k-1}) \arrow[r, "s"] & \tilde{H}_{k-1}(S) \arrow[r] & \tilde{H}_{k-1}(V) \end{tikzcd}$$

where all the horizontal homomorphisms are isomorphisms.

Set $\mathcal{W} := \{W_0, W_1\}$. This is an open cover of S^k and we have the following commutative diagram of chain complexes with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \tilde{S}_*(Y) \arrow[r] \arrow[d, "\tilde{S}_*(\pi)"'] & \tilde{S}_*(W_0) \oplus \tilde{S}_*(W_1) \arrow[r] \arrow[d, "\tilde{S}_*(\pi)\oplus\tilde{S}_*(\pi)"] & \tilde{S}_*^{\mathcal{W}}(S^k) \arrow[r] \arrow[d, "\tilde{S}_*(\pi)"] & 0 \\ 0 \arrow[r] & \tilde{S}_*(V) \arrow[r] & \tilde{S}_*(U_0) \oplus \tilde{S}_*(U_1) \arrow[r] & \tilde{S}_*^{\mathcal{U}}(\mathbf{RP}^k) \arrow[r] & 0 \end{tikzcd}$$

These exact sequences lead to long exact sequences of their homologies. Since W_0 and V are homotopy equivalent to S^{k-1}, their k-th homology are trivial. As W_1 is a disjoint union of contractible spaces we also have that $\tilde{H}_k(W_1)$ and $\tilde{H}_{k-1}(W_1)$ are trivial. We thus get a commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \tilde{H}_k(S^k) \arrow[r] \arrow[d, "\tilde{H}_k(\pi)"] & \tilde{H}_{k-1}(Y) \arrow[r] \arrow[d, "\tilde{H}_{k-1}(\pi)"] & \tilde{H}_{k-1}(W_0) \arrow[d, "\tilde{H}_{k-1}(\pi)"] \\ 0 \arrow[r] & \tilde{H}_k(\mathbf{RP}^k) \arrow[r] & \tilde{H}_{k-1}(V) \arrow[r] & \tilde{H}_{k-1}(U_0) \end{tikzcd}$$

Note that the bottom sequence is the sequence from page 102. We can thus rewrite it as:

$$\begin{tikzcd} 0 \arrow[r] & \tilde{H}_k(S^k) \arrow[r] \arrow[d, "\tilde{H}_k(\pi)"] & \tilde{H}_{k-1}(Y) \arrow[r] \arrow[d] & \tilde{H}_{k-1}(W_0) \arrow[d] \\ 0 \arrow[r] & \tilde{H}_k(\mathbf{RP}^k) \arrow[r, "\partial"] & \tilde{H}_{k-1}(S^{k-1}) \arrow[r, "\tilde{H}_{k-1}(\pi)"] & \tilde{H}_{k-1}(\mathbf{RP}^{k-1}) \end{tikzcd}$$

If we identify the appropriate groups with $\mathbf{Z}$ and use Exercise 8.12.6, the above commutative diagram becomes:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{H}_k(S^k) \arrow[r] \arrow[d, "\tilde{H}_k(\pi)"] & \mathbf{Z} \oplus \mathbf{Z} \arrow[r, "1+1"] \arrow[d, "1+(-1)^k"] & \mathbf{Z} \arrow[d] \\
0 \arrow[r] & \tilde{H}_k(\mathbf{RP}^k) \arrow[r] & \mathbf{Z} \arrow[r] & \tilde{H}_{k-1}(\mathbf{RP}^{k-1})
\end{tikzcd}$$

In this diagram, the generator of $\tilde{H}_k(S^k)$ is mapped to an element $(1,-1) \in \mathbf{Z}\oplus\mathbf{Z}$ via the horizontal homomorphism. It follows that we get a commutative diagram:

$$\begin{tikzcd}
\mathbf{Z} = \tilde{H}_k(S^k) \arrow[r, "\tilde{H}_k(\pi)"] \arrow[rr, bend right, "1+(-1)^{k+1}"'] & \tilde{H}_k(\mathbf{RP}^k) \arrow[r, "\partial"] & \tilde{H}_{k-1}(S^{k-1}) = \mathbf{Z}
\end{tikzcd}$$

We can thus conclude:

8.12.7 Corollary. *If k is odd, then the image of the following composition is given by the subgroup generated by 2:*

$$\mathbf{Z} = \tilde{H}_k(S^k) \xrightarrow{\tilde{H}_k(\pi)} \tilde{H}_k(\mathbf{RP}^k) \xrightarrow{\partial} \tilde{H}_k(S^{k-1}) = \mathbf{Z}$$

In particular the homomorphism $\tilde{H}_k(\pi) : \tilde{H}_k(S^k) \longrightarrow \tilde{H}_k(\mathbf{RP}^k)$ is a monomorphism.

Recall that we have an exact sequence:

$$\begin{tikzcd}
 & 0 \arrow[r] & 0 \arrow[r] & \tilde{H}_k(\mathbf{RP}^k) \arrow[dll, "\partial"'] \\
 & \tilde{H}_k(S^{k-1}) \arrow[r, "\tilde{H}_{k-1}(\pi)"'] & \tilde{H}_{k-1}(\mathbf{RP}^{k-1}) \arrow[r, "\tilde{H}_{k-1}(i)"] & \tilde{H}_{k-1}(\mathbf{RP}^k) \arrow[dll] \\
 & 0 & &
\end{tikzcd}$$

Using this exact sequence and Corollary 8.12.7 we get:

8.12.8 Corollary. *Let $k \geq 1$.*

(1) If k is even, then $\tilde{H}_k(\mathbf{RP}^k) = 0$.

(2) If k is odd, then $\partial : \tilde{H}_k(\mathbf{RP}^k) \longrightarrow \tilde{H}_{k-1}(S^{k-1})$ is an isomorphism. In particular $\tilde{H}_k(\mathbf{RP}^k)$ is isomorphic to $\mathbf{Z}$.

Proof. The proof is by induction on k. Case $k = 1$ is left as an exercise (use the above exact sequence and observe that $\tilde{H}_0(\mathbf{RP}^0) = \tilde{H}_0(\mathbf{RP}^1) = 0$).

Assume that $k > 1$ and that the lemma is true for projective spaces whose dimension is less than k. If k is even, then $k-1$ is odd and and hence by Corollary 8.12.7, $\tilde{H}_{k-1}(\pi) : \tilde{H}_k(S^{k-1}) \longrightarrow \tilde{H}_{k-1}(\mathbf{RP}^{k-1})$ is a monomorphism. Its kernel $\tilde{H}_k(\mathbf{RP}^k)$ is therefore trivial.

Assume that k is odd. Thus $k-1$ is even and by the inductive assumption $\tilde{H}_{k-1}(\mathbf{RP}^{k-1}) = 0$. It then follows that the boundary homomorphism: $\partial : \tilde{H}_k(\mathbf{RP}^k) \longrightarrow \tilde{H}_{k-1}(S^{k-1})$ is an isomorphism. $\square$

We can now identify explicitly the image of:

8.12.9 Lemma. *Let $k \geq 1$ be odd. Then the image of:*

$$\mathbf{Z} = \tilde{H}_k(S^k) \xrightarrow{\tilde{H}_k(\pi)} \tilde{H}_k(\mathbf{RP}^k) = \mathbf{Z}$$

is the subgroup generated by 2.

Proof. Recall that since k is odd, we have a commutative diagram:

$$\begin{tikzcd} \mathbf{Z} = \tilde{H}_k(S^k) \arrow[r, "\tilde{H}_k(\pi)"] \arrow[rr, bend right, "2"'] & \tilde{H}_k(\mathbf{RP}^k) \arrow[r, "\partial"] & \tilde{H}_{k-1}(S^{k-1}) = \mathbf{Z} \end{tikzcd}$$

The lemma now is a consequence of the fact that the boundary homomorphism $\partial : \tilde{H}_k(\mathbf{RP}^k) \longrightarrow \tilde{H}_{k-1}(S^{k-1})$ is an isomorphism. $\square$

We are now ready to prove:

Proof of Theorem 8.12.1. By Lemma 8.12.2 and Corollary 8.12.8 we already know that the theorem is true when $n \geq k \geq 1$. It remains to prove the theorem for $n < k$. This will be done by induction on k. The case $k = 1$ is clear. Let $k > 1$ and assume that the theorem is true for projective spaces whose dimension is smaller than k. Since $\tilde{H}_n(i) : \tilde{H}_n(\mathbf{RP}^{k-1}) \longrightarrow \tilde{H}_n(\mathbf{RP}^k)$ is an isomorphism for $n \neq k$ and $n \neq k-1$, by the inductive assumption we then know that the theorem is true for $n \leq k-2$. It remains to show that the theorem is also true for $n = k-1$. Recall that there is an exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & \tilde{H}_k(\mathbf{RP}^k) \arrow[dll, "\partial"] \\ \tilde{H}_{k-1}(S^{k-1}) \arrow[r, "\tilde{H}_{k-1}(\pi)"'] & \tilde{H}_{k-1}(\mathbf{RP}^{k-1}) \arrow[r, "\tilde{H}_{k-1}(i)"] & \tilde{H}_{k-1}(\mathbf{RP}^k) \arrow[dll] \\ 0 & & \end{tikzcd}$$

We now consider two cases. First let k be odd, and hence $k-1$ is even. Thus $\tilde{H}_{k-1}(\mathbf{RP}^{k-1}) = 0$ and from the exactness of the above sequence it follows that $\tilde{H}_{k-1}(\mathbf{RP}^k) = 0$. In the case k is even, then $k-1$ is odd and therefore, according to Lemma 8.12.9, the image of $\tilde{H}_{k-1}(\pi) : \tilde{H}_{k-1}(S^{k-1}) \longrightarrow \tilde{H}_{k-1}(\mathbf{RP}^{k-1}) = \mathbf{Z}$ is is the subgroup generated by the element 2. It then follows from this exact sequence that $\tilde{H}_{k-1}(\mathbf{RP}^k) = \mathbf{Z}/2$. That finishes the proof of the theorem. □

8.13 Homology of projective spaces

So far we have calculated the integral homology groups of the projective spaces. In this section we are going to use calculation to present $\tilde{H}_n(\mathbf{RP}^k, G)$ for any group G. We start with rephrasing the algebraic universal coefficient theorem (Theorem 6.6.1), in terms of the homology of topological spaces:

8.13.1 Theorem. *Let X be a space and G be a group. Then:*

$$\tilde{H}_n(X, G) = \begin{cases} \tilde{H}_0(X) \otimes G & \text{if } n = 0 \\ \big(\tilde{H}_n(X) \otimes G\big) \oplus \operatorname{Tor}_1^{\mathbf{Z}}(\tilde{H}_{n-1}(X), G) & \text{if } n > 0 \end{cases}$$

We can now use the above theorem and Theorem 8.12.1 to get:

8.13.2 Theorem. *Let $k \geq 1$. Then:*

(1) if k is even

$$\tilde{H}_n(\mathbf{RP}^k, G) = \begin{cases} 0 & \text{if } n = 0 \text{ or } n > k \\ \mathbf{Z}/2 \otimes_{\mathbf{Z}} G & \text{if } n \text{ is odd and } n \leq k \\ \operatorname{Tor}_1^{\mathbf{Z}}(\mathbf{Z}/2, G) & \text{if } n \text{ is even and } n \leq k \end{cases}$$

(2) if k is odd

$$\tilde{H}_n(\mathbf{RP}^k, G) = \begin{cases} 0 & \text{if } n = 0 \text{ or } n > k \\ \mathbf{Z}/2 \otimes_{\mathbf{Z}} G & \text{if } n \text{ is odd and } n < k \\ G & \text{if } n = k \\ \operatorname{Tor}_1^{\mathbf{Z}}(\mathbf{Z}/2, G) & \text{if } n \text{ is even and } n \leq k \end{cases}$$

For example in the case $G = \mathbf{Z}/2$ or $G = \mathbf{Q}$ we get:

8.13.3 Corollary. *Let $k \geq 1$. Then:*

$$\tilde{H}_n(\mathbf{RP}^k, \mathbf{Z}/2) = \begin{cases} 0 & \text{if } n = 0 \text{ or } n > k \\ \mathbf{Z}/2 & \text{if } n \geq k > 0 \end{cases}$$

$$\tilde{H}_n(\mathbf{RP}^k, \mathbf{Q}) = \begin{cases} 0 & \text{if } k \text{ is even or } n \neq k \\ \mathbf{Q} & \text{if } k \text{ is odd and } n = k \end{cases}$$

8.13.4 Excercise. Calculate the following homology groups:

$$\tilde{H}_n(S^1 \times S^1) \qquad \tilde{H}_n(S^1 \times S^1, \mathbf{Z}/2) \qquad \tilde{H}_n(S^1 \times S^1, \mathbf{Q})$$

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