Counting permutations by their alternating runs

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Abstract

We find a formula for the number of permutations of $[n]$ that have exactly s runs up and down. The formula is at once terminating, asymptotic, and exact. The asymptotic series is valid for $n \to \infty$, uniformly for $s \leqslant (1 - \epsilon)n / \log n$ ($\epsilon > 0$).
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1. Introduction

We will say that a *run* of a permutation σ is a maximal interval of consecutive arguments of σ on which the values of σ are monotonic. If the values of σ increase on the interval then we speak of a *run up*, else a *run down*. Throughout this paper we will use the unqualified term *run* to mean either a run up or a run down. These runs have been called *sequences* by some other authors, and have been called *alternating runs* by others. For example, the permutation

(723851469)

has four runs, viz. 72, 238, 851, 1469. We let $P(n, s)$ denote the number of permutations of n letters that have exactly s runs. Here are the first few values of $P(n, s)$:

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$n \backslash s$	1	2	3	4
2	2			
3	2	4		
4	2	12	10	
5	2	28	58	32

There is a large literature devoted to this $P(n, s)$. Although a number of recurrences and generating functions are known, it does not seem to have been noticed that an interesting exact formula of the kind we present in this paper exists. Carlitz [6] has derived an exact formula for $P(n, s)$, but that one is not at the same time an asymptotic formula. We comment further on Carlitz's formula in a moment.

André was the first to study [1] the runs up and down of permutations, and the fundamental recurrence, (2) below, is due to him. His paper includes a table of $P(n, s)$ through $n = 8$, with one error in the final row. A great deal of information about $P(n, s)$ is found in vol. 3 of [9] (see particularly Ex. 15, 16 of Section 5.1.3).

The history of generating functions in this problem is complex. Comtet [8, p. 260] devotes an extended exercise to the topic. The two variable generating function given there, however, is incorrect. Carlitz [5–7] visited this subject several times. In [5] he gives a two-variable generating function

$$\sum_{n=2}^{\infty} \frac{z^n}{n!} \left(1 - x^2\right)^{-n/2} \sum_{s=1}^{n-1} P(n+1, s) x^{n-s} = \frac{(1-x)((1-x^2)^{1/2} + \sin(z))^2}{(1+x)(x - \cos(z))^2},$$

and in [6] he finds an explicit formula for $P(n, s)$ and information about an associated polynomial sequence. There is something wrong with the final formulas of this latter work, however; these formulas suggest $P(8, s) = 0, 2, 250, 2516, 7060, 7562, 2770$; whereas, in fact, $P(8, s) = 2, 252, 2766, 9576, 14622, 10332, 2770$. (Empirically, his formula always gives the right value for $P(n, n-1)$.) Further evidence that something is amiss concerns the auxiliary quantity $\overline{K}_{n,j}$; the summation formula given for this quantity does not give the values displayed in the table.

A correct generating function appears in the discussion accompanying sequence A059427 of [10]. This one is due to Emeric Deutsch and Ira Gessel, who sent it to Neil Sloane in December of 2004. A correct generating function is also in Stanley [11], who used an observation of Miklós Bóna to connect the sequence that we study here with $a_k(n)$, the number of n-permutations the length of whose longest alternating subsequence is k.

Bóna and Ehrenborg [4] have proven log-concavity: $P(n, s)^2 \geqslant P(n, s-1) P(n, s+1)$. In the later book [3], the stronger assertion, that $P_n(x) \stackrel{\text{def}}{=} \sum_s P(n, s) x^s$ has all its roots real and negative, is made. A proof of this can be based on the relation

$$P_n(x) = \left(x - x^3\right) P'_{n-1}(x) + \left((n-2)x^2 + 2x\right) P_{n-1}(x),$$

which itself is a consequence of the basic recursion (2). This implies, once it is established that the variance becomes infinite with n, that the numbers $P(n, s)$ satisfy a central limit theorem. (That is, are asymptotically normal.) Due to log-concavity, one may deduce (see [2, Theorem 4]) a local limit theorem. This leads to an asymptotic formula for $P(n, s)$ for s in a different, and disjoint, range than in our Theorem 2.

2. A new approach

Our approach to this problem differs from previous studies in that we concentrate on the column generating functions $u_s(x)$, defined for each fixed $s \geqslant 1$ by

$$u_s(x) = \sum_{n \geqslant 2} P(n, s)x^n,$$

whereas most earlier work has dealt with generating functions for fixed n. By finding the form of these generating functions we will be able to exhibit a formula for $P(n, s)$ which is simultaneously

- exact, and
- terminating, and
- asymptotic, for $n \to \infty$ and $s \leqslant (1 - \epsilon)n/\log n$.

To our knowledge, the asymptotic behavior of the $P(n, s)$ has not been previously explored.

The formula that we will find is of the form

$$P(n, s) = \frac{s^n}{2^{s-2}} - \frac{(s-1)^n}{2^{s-4}} + \psi_2(n, s)(s-2)^n + \cdots + \psi_{s-1}(n, s) \quad (n \geqslant 2), \tag{1}$$

in which each $\psi_i(n, s)$ is a polynomial in n whose degree in n is $\lfloor i/2 \rfloor$.

Here is an outline of the rest of the paper. In Section 3 we will find the generating functions $u_s(x) = \sum_n P(n, s)x^n$, as rational functions. Since the denominators will appear in completely factored form, we can write out, in Section 4, a formula for $P(n, s)$ of the type described above.

Interestingly, the formula will be, in that section, uniquely determined except for the coefficient of the leading term! That is, we will show in that section, that for fixed s we have $P(n, s) = K(s)s^n + \cdots$, but $K(s)$ will be, for the moment, unknown.

In Section 5 we begin the task of determining the multiplicative factor $K(s)$. Surprisingly, although the tools that will have been used up to that point will be entirely analytical in nature, the determination of $K(s)$ will be done by an "almost-bijection." We will show that the product $2^{s-2} \times P(n, s)$ is, for $s \leqslant (1 - \epsilon)n/\log n$, asymptotic to the number of s-tuples of pairwise-disjoint subsets of $[n]$, each of cardinality $\geqslant 2$, and whose union equals $[n]$; the asymptotic behavior of the latter is easily found.

The combination of the former analytical results and the latter bijective argument results in the complete formula for $P(n, s)$.

3. Finding the $u_s(x)$ functions

The recurrence formula for the numbers $P(n, s)$ is well known and is due to André [1],

$$P(n, s) = s P(n - 1, s) + 2 P(n - 1, s - 1) + (n - s)P(n - 1, s - 2) \quad (n \geqslant 3), \tag{2}$$

with $P(2, s) = 2\delta_{s,1}$. From this recurrence one finds easily a recurrence for the generating functions $u_s(x) \stackrel{\text{def}}{=} \sum_n P(n, s)x^n$, viz.

$$(1 - sx)u_s(x) = 2xu_{s-1}(x) + x^2 u'_{s-2}(x) - (s - 1)xu_{s-2}(x) \quad (s \geqslant 2), \tag{3}$$

with $u_1(x) = 2x^2/(1 - x), u_0(x) = 0$. The next three of these functions are

$$u_2(x) = \frac{4x^3}{(1-x)(1-2x)},$$

$$u_3(x) = \frac{2x^4(5-6x)}{(1-3x)(1-2x)(1-x)^2},$$

$$u_4(x) = \frac{4x^5(8-29x+24x^2)}{(1-4x)(1-3x)(1-2x)^2(1-x)^2}.$$

We will find the general form of these functions, and from that will follow the desired formulas for $P(n,s)$.

Theorem 1. *We have, for each $s = 1, 2, 3, \ldots$,*

$$u_s(x) = \frac{\Phi_s(x)}{(1-sx)(1-(s-1)x)(1-(s-2)x)^2(1-(s-3)x)^2 \cdots (1-x)^{\lfloor (s+1)/2 \rfloor}}, \quad (4)$$

where $\Phi_s(x)$ is a polynomial of degree $1 + \lceil \frac{s(s+2)}{4} \rceil$. The degree of the denominator is $\lceil \frac{s(s+2)}{4} \rceil$, which is exactly 1 less than the degree of the numerator, for all $s \geq 1$.

3.1. Proof of Theorem 1

The proof is by a straightforward, though tedious, substitution of the form (4) into the recurrence (3) to find a recurrence for the numerator polynomials $\Phi_s(x)$. This will establish that they are indeed polynomials and will provide the claimed degree estimates. We will do this by putting every term over the common denominator

$$\Delta_s(x) = (1-sx)\big(1-(s-1)x\big)\big(1-(s-2)x\big)^2\big(1-(s-3)x\big)^2 \cdots (1-x)^{\lfloor (s+1)/2 \rfloor}$$

$$\overset{\text{def}}{=} \prod_{i=0}^{s-1} \big(1-(s-i)x\big)^{\epsilon_i},$$

where we have written $\{\epsilon_i\}_{i \geq 0} = \{1, 1, 2, 2, 3, 3, 4, 4, \ldots\}$.

If we substitute the form (4) into the recurrence (3) we obtain

$$u_s(x) = \frac{\Phi_s(x)}{\Delta_s(x)} = \frac{2xu_{s-1}(x)}{(1-sx)} + \frac{x^2 u'_{s-2}(x)}{(1-sx)} - \frac{(s-1)xu_{s-2}(x)}{(1-sx)}$$

$$= \frac{2x\Phi_{s-1}(x)}{(1-sx)\Delta_{s-1}(x)} + \frac{x^2\Phi'_{s-2}(x)}{(1-sx)\Delta_{s-2}} - \frac{x^2\Phi_{s-2}}{(1-sx)\Delta_{s-2}} \frac{\Delta'_{s-2}(x)}{\Delta_{s-2}(x)}$$

$$\quad - \frac{(s-1)x\Phi_{s-2}(x)}{(1-sx)\Delta_{s-2}(x)}$$

$$= \frac{1}{\Delta_s(x)} \left\{ \frac{2x\Phi_{s-1}(x)\Delta_s(x)}{(1-sx)\Delta_{s-1}(x)} + \frac{x^2\Phi'_{s-2}(x)\Delta_s(x)}{(1-sx)\Delta_{s-2}} - \frac{x^2\Phi_{s-2}(x)\Delta_s(x)}{(1-sx)\Delta_{s-2}} \frac{\Delta'_{s-2}(x)}{\Delta_{s-2}(x)} \right.$$

$$\left. \quad - \frac{(s-1)x\Phi_{s-2}(x)\Delta_s(x)}{(1-sx)\Delta_{s-2}(x)} \right\}. \quad (5)$$

Hence we have found the recurrence that the numerators $\Phi_s(x)$ satisfy, and it is

$$\Phi_s = \Phi_{s-1} \frac{2x\Delta_s}{(1-sx)\Delta_{s-1}} + \left(x^2\Phi'_{s-2} - (s-1)x\Phi_{s-2}\right)\frac{\Delta_s}{(1-sx)\Delta_{s-2}}$$

$$- \Phi_{s-2}\frac{x^2\Delta_s}{(1-sx)\Delta_{s-2}}\frac{\Delta'_{s-2}}{\Delta_{s-2}}. \tag{6}$$

We claim that the coefficients of the Φ's above are all polynomials, and we will find their degrees.

Consider the ratio

$$\frac{\Delta_s(x)}{(1-sx)\Delta_{s-1}(x)} = \frac{\prod_{j=0}^{s-1}(1-(s-j)x)^{\epsilon_j}}{(1-sx)\prod_{j=0}^{s-2}(1-(s-1-j)x)^{\epsilon_j}}$$

$$= \frac{\prod_{j=0}^{s-1}(1-(s-j)x)^{\epsilon_j}}{(1-sx)\prod_{j=1}^{s-1}(1-(s-j)x)^{\epsilon_{j-1}}}$$

$$= \prod_{j=1}^{s-1}\left(1-(s-j)x\right)^{\epsilon_j-\epsilon_{j-1}} = \prod_{j \text{ even}; \ 2\leqslant j\leqslant s-1}\left(1-(s-j)x\right),$$

which is a polynomial of degree $\lfloor(s-1)/2\rfloor$.

It follows that

$$\frac{\Delta_s(x)}{(1-sx)\Delta_{s-2}(x)} = \left(\frac{\Delta_s(x)}{(1-sx)\Delta_{s-1}(x)}\right)\left(\frac{\Delta_{s-1}(x)}{(1-(s-1)x)\Delta_{s-2}(x)}\right)(1-(s-1)x)$$

$$= \prod_{j \text{ even}; \ 2\leqslant j\leqslant s-1}\left(1-(s-j)x\right) \prod_{j \text{ even}; \ 0\leqslant j\leqslant s-2}\left(1-(s-1-j)x\right)$$

$$= \prod_{j \text{ even}; \ 2\leqslant j\leqslant s-1}\left(1-(s-j)x\right) \prod_{j \text{ odd}; \ 1\leqslant j\leqslant s-1}\left(1-(s-j)x\right)$$

$$= \prod_{j=1}^{s-1}\left(1-(s-j)x\right),$$

is a polynomial in x of degree $s-1$.

Next, since

$$\frac{\Delta'_{s-2}(x)}{\Delta_{s-2}(x)} = -\sum_{j=2}^{s-1}\frac{\epsilon_{j-2}(s-j)}{1-(s-j)x},$$

we have

$$\frac{x^2\Phi_{s-2}(x)\Delta_s(x)}{(1-sx)\Delta_{s-2}}\frac{\Delta'_{s-2}(x)}{\Delta_{s-2}(x)} = x^2\Phi_{s-2}(x)\left(\prod_{j=1}^{s-1}(1-(s-j)x)\right)\left(\sum_{j=2}^{s-1}\frac{-\epsilon_{j-2}(s-j)}{1-(s-j)x}\right).$$

If we make the inductive assumption that each Φ_i for $i < s$ is a polynomial in x of degree $d(i)$, then this last member is a polynomial in x of degree $2 + d(s-2) + s - 2 = d(s-2) + s$.

We have now shown that all of the terms on the right side of (6) are polynomials in x. Their respective degrees are

$$d(s-1) + 1 + \lfloor(s-1)/2\rfloor, \quad d(s-2) + s, \quad d(s-2) + s, \quad d(s-2) + s.$$

Hence $\Phi_s(x)$ is indeed a polynomial in x and its degree is

$$d(s) = \max\big(d(s-1) + \lfloor (s+1)/2 \rfloor, d(s-2) + s\big),$$

with $d(2) = 3$ and $d(3) = 5$.

It is remarkable that this difference equation has a simple solution. Its solution is

$$d(s) = 1 + \left\lceil \frac{s(s+2)}{4} \right\rceil,$$

as can easily be checked, and in fact all four terms inside the braces in (5) have the same degree! This completes the proof of the theorem.

4. The formula for $P(n,s)$

From the partial fraction expansion of (4) we find at once that

$$P(n,s) = \psi_0(n,s)s^n + \psi_1(n,s)(s-1)^n + \psi_2(n,s)(s-2)^n + \cdots + \psi_{s-1}(n,s)$$
$$(n \geq 2), \tag{7}$$

where each $\psi_i(n,s)$ is a polynomial in n of degree at most $\lfloor i/2 \rfloor$, and it remains to find these polynomials. We give three methods of doing this: a method of undetermined coefficients, a differential recurrence formula, and finally, a formula of Richard Stanley [11].

4.1. Finding the ψ_i's by undetermined coefficients

Substitute (7) into the recurrence (2) and match the coefficients of each term $(s-i)^n$. The result of this substitution is that the ψ_i's satisfy the recurrence

$$(s-i)\psi_i(n,s) = s\psi_i(n-1,s) + 2\psi_{i-1}(n-1,s-1) + (n-s)\psi_{i-2}(n-1,s-2). \tag{8}$$

It should be noted that even if ψ_{i-1} and ψ_{i-2} are known, the unknown ψ_i appears in two places in this recurrence, so we must solve an inhomogeneous difference equation for each i.

However, we can just assume a solution in the form of a polynomial in n of degree $\lfloor i/2 \rfloor$ and solve for the coefficients of that polynomial. We can begin with $\psi_{-1}(n,s) = 0$ and $\psi_0(n,s) = K(s)$ (since ψ_0 is of degree zero in n) where K is to be determined. We then find that

$$\psi_1(n,s) = -2K(s-1), \qquad \psi_2(n,s) = \frac{1}{4}K(s-2)(s+8-2n),$$

$$\psi_3(n,s) = \frac{1}{2}K(s-3)(2n-s-3),$$

$$\psi_4(n,s) = \frac{1}{32}K(s-4)\big(4n^2 - 4n(s+8) + s^2 + 15s + 32\big).$$

This is as far as we can go without having determined the function $K(s)$, which will be done below in Section 5. However, if we anticipate the result of that section, which is that $K(s) = 2^{-(s-2)}$, then, for example, for $s = 4$ we would find the exact formula

$$P(n,4) = 4^{n-1} - 3^n + (6-n)2^{n-1} + (2n-7) \quad (n \geq 2).$$

4.2. Finding the ψ_i's recursively

Another method for finding the ψ_i's involves solving the recurrence directly. This leads to a surprisingly elegant differential recurrence, as we will now see. First we need the following lemma about the polynomial solutions of first order inhomogeneous difference equations.

Lemma 1. *Let $C \neq 1$, and let f be a polynomial. Then the difference equation $y_{n+1} = Cy_n + f(n)$ has a unique polynomial solution, namely*

$$y_n = -C^{n-1} f\left(x\frac{d}{dx}\right)\left(\frac{x^n}{1-x}\right)\bigg|_{x=1/C}. \tag{9}$$

For example, the difference equation $y_{n+1} = 3y_n + 3n + 2$ has the unique polynomial solution

$$y_n = -3^{n-1}\left(3x\frac{d}{dx} + 2\right)\left(\frac{x^n}{1-x}\right)\bigg|_{x=1/3} = -\frac{3n}{2} - \frac{7}{4}.$$

To prove the lemma we note first that the general solution of $y_{n+1} = Cy_n + f(n)$ is evidently

$$y_n = C^n y_0 + \sum_{j=0}^{n-1} C^{n-j-1} f(j) \quad (n = 0, 1, 2, \ldots),$$

and we need to discover when this is a polynomial in n. Now suppose that $f(n) = \sum_k \alpha_k n^k$. Then we have

$$y_n = C^n y_0 + C^{n-1} \sum_k \alpha_k \sum_{j=0}^{n-1} C^{-j} j^k.$$

But it is easy to check by induction that

$$\sum_{j=0}^{n-1} j^k x^j = \left(x\frac{d}{dx}\right)^k \left(\frac{1-x^n}{1-x}\right) = x^n \frac{R_k(x, n)}{(x-1)^{k+1}} + \frac{Q_k(x)}{(x-1)^{k+1}}, \tag{10}$$

where R_k is a polynomial in x of degree k and is also a polynomial in n of degree k, and Q_k is a polynomial in x of degree k. Consequently the general solution is

$$y_n = C^n y_0 + C^{n-1} \sum_k \alpha_k \frac{Q_k(C^{-1})}{(C^{-1}-1)^{k+1}} + C^{-1} \sum_k \alpha_k \frac{R_k(C^{-1}, n)}{(C^{-1}-1)^{k+1}}.$$

Since $C \neq 1$, this will be a polynomial in n if and only if

$$y_0 = -C^{-1} \sum_k \alpha_k \frac{Q_k(C^{-1})}{(C^{-1}-1)^{k+1}},$$

and if that condition is satisfied the unique polynomial solution will be

$$y_n = \frac{1}{C} \sum_k \alpha_k \frac{R_k(C^{-1}, n)}{(C^{-1}-1)^{k+1}}.$$

We can simplify the form of this answer by recalling that, from (10) we have

$$\frac{R_k(x, n)}{(x-1)^{k+1}} = -x^{-n}\left(x\frac{d}{dx}\right)^k\left(\frac{x^n}{1-x}\right),$$

and therefore we can cast the unique polynomial solution in the more pleasing form (9).

Hence we have the following procedure for calculating the ψ_i's. For each $i = 1, 2, \ldots$ we do

(1) Suppose ψ_{i-2} and ψ_{i-1} are known.
(2) Define the polynomial

$$f(n) = \frac{2}{s-i} \psi_{i-1}(n, s-1) + \frac{n+1-s}{s-i} \psi_{i-2}(n, s-2),$$

and put $C = s/(s-i)$.
(3) Then

$$\psi_i(n, s) = -C^{n-1} f\left(x\frac{d}{dx}\right)\left(\frac{x^n}{1-x}\right)\Big|_{x=1/C}.$$

4.3. Finding the ψ_i's from Stanley's formula

In [11] Richard Stanley has given an exact formula for $P(n, s)$, viz.

$$P(n, s) = \sum_{\ell=0}^{s} (-1)^{s-\ell} \frac{z_{s-\ell}}{2^{\ell-1}} \sum_{\substack{r+2m \leqslant \ell \\ r \equiv \ell \bmod 2}} (-2)^m \binom{\ell - m}{(\ell+r)/2} \binom{n}{m} r^n,$$

where $z_0 = 2$ and all other z_i's are 4. Evidently this contains an implicit formula for our ψ's.

5. The factor $K(s)$

We have now described the formula for $P(n, s)$ completely except for the multiplicative factor $K(s)$. It remains to show that $K(s) = 2^{-(s-2)}$. For this, it would suffice to prove the next theorem for fixed s and $n \to \infty$; since the proof is applicable to a larger range of s, we state it in that manner:

Theorem 2. *Let $\epsilon > 0$, and $\{(n, s)\}$ be an infinite sequence of pairs such that $n \to \infty$ and $s \leqslant (1+\epsilon)^{-1} n/\log n$. Then,*

$$P(n, s) \sim \frac{1}{2^{s-2}} s^n. \tag{11}$$

This theorem can be deduced from Stanley's [11] exact formula for $P(n, s)$. Our proof is an interesting alternate approach, based on an "almost bijection," which we think is also worthy of presentation.

5.1. Proof of Theorem 2

To fix ideas, we will do this by showing that the number $\hat{P}(n, s)$ of permutations of n letters, with s runs, *the first of which is a run up*, is $\sim s^n/2^{s-1}$. Evidently the number for which the first run is down will be the same, and the desired result will follow. Henceforth we will always assume that the first run is a run up. There are two steps to the proof. In the first step, we show that the set of permutations counted by $\hat{P}(n, s)$ can be put into bijection with s-tuples of subsets $(S_1, \ldots, S_s)$ (each $S_i \subseteq [n]$) satisfying certain properties. In the second part of the proof, we

introduce a function called Φ whose domain is the Cartesian product of these s-tuples with a set of cardinality 2^{s-1}, and whose range is a set of size s^n. We prove that this function Φ is an *injection*. Although we have no succinct description of the image of this injection, we are able to show that for (n, s) in the range hypothesized by the theorem the image is asymptotically all of the range set.

5.2. First part of the proof

Let $\Pi(n, s)$ be the set of all n-permutations with s runs up and down, the first of which is up. Let $\tilde{\Pi}(n, s)$ be the collection of all s-tuples $(S_1, \ldots, S_s)$ of nonempty subsets of $[n]$ which are *almost* pairwise disjoint, in that

$$|S_i \cap S_j| = \begin{cases} 1, & \text{if } j = i + 1 \text{ and } 1 \leqslant i < s; \\ 0, & \text{else.} \end{cases} \tag{12}$$

Further we require that

$$|S_i| \geqslant 2, \quad \forall i, \tag{13}$$

and that

$$\begin{aligned} \max(S_i) &= \max(S_{i+1}) \in S_i \cap S_{i+1} \quad (\forall \text{ odd } i), \\ \min(S_i) &= \min(S_{i+1}) \in S_i \cap S_{i+1} \quad (\forall \text{ even } i). \end{aligned} \tag{14}$$

Lemma 2. *The number of s-tuples of subsets of $[n]$ that satisfy (12)–(14) is equal to the number of permutations of $[n]$ with s runs, the first of which is up.*

Indeed to reconstruct the permutation from the s-tuple of sets, we first sort each of the sets, the first in increasing order, the second decreasing, etc., then merge the sets, and finally delete one element of each of the adjacent duplicates that appear.

Hence it suffices to show that the number of s-tuples of subsets that satisfy (12)–(14) is $\sim s^n / 2^{s-1}$.

5.3. Defining the function Φ

By a *choice sequence* $\mathbf{h} = (h_1, \ldots, h_{s-1})$ we mean an $s - 1$-tuple where each h_i is either equal to i or to $i + 1$. The set of all such choice sequences will be H_s. The function to be constructed is a mapping

$$\Phi : H_s \times \tilde{\Pi}(n, s) \to \big\{ (T_1, T_2, \ldots, T_s) \colon \forall i, T_i \subseteq [n] \big\}.$$

Let $\mathbf{h} \in H_s$, and let $(S_1, \ldots, S_s)$ be a family of subsets satisfying (12)–(14). For each $i = 1, \ldots, s - 1$, let e_i be the unique element that belongs to $S_i \cap S_{i+1}$. These e_i's are all different, since $e_i = e_j$ with $i < j$ would imply that $S_i \cap S_{j+1}$ is nonempty, contradicting (12). Perform the following $s - 1$ delete operations: for each $i = 1, \ldots, s - 1$, delete the element e_i from the set S_{h_i}. The resulting s-tuple of sets remaining after these deletions is, by definition, $\Phi(\mathbf{h}, (S_1, \ldots, S_s))$.

The image of this mapping does not include all s-tuples of sets, as the following lemma shows.

Lemma 3. *If $(T_1, \ldots, T_s)$ is in the image of Φ then*

(1) *the T_i's are pairwise disjoint, and*
(2) *the union of the T_i's is $[n]$.*

It is possible for some of the T_i's to be empty. We remark that the number of s-tuples $(T_1, \ldots, T_s)$ in which the T_i's are pairwise disjoint and whose union is $[n]$ is s^n.

5.4. The mapping Φ is injective

The way we prove this assertion is to give a *reconstruction* algorithm. The algorithm begins with an s-tuple $(T_1, \ldots, T_s)$ of subsets which putatively belongs to the image of Φ. It attempts to reconstruct the preimage. It will be clear from the algorithm that the preimage can be only one thing, if it exists at all. There is one "early exit" point in the algorithm where the search for a preimage is abandoned, because it obviously does not exist. If the algorithm executes all the way to finish, then it will have found the only possible candidate for a preimage. However, it is still possible that the s-tuple of sets found at the end will not satisfy one of the required conditions (12)–(14).

Lemma 4. *The mapping Φ is injective.*

Proof. Before presenting the reconstruction algorithm, we need to verify a claim: if the T_i's are in the image of Φ, then

$$\max(T_i \cup T_{i+1}) = \max(S_i \cup S_{i+1}) \quad \text{for } i \text{ odd,}$$

and similarly for the minima with i even. Indeed, $T_i \subseteq S_i$, so certainly the right side is less than or equal to the left. However, the element $e_i = \max(S_i, S_{i+})$ is deleted from at most one set (since the e_i's are distinct), and so it remains in at least one of T_i or T_{i+1}. The claim is justified. Let now $(T_1, \ldots, T_s)$ be an s-tuple of pairwise disjoint (possibly empty) sets whose union is $[n]$. Here is the reconstruction algorithm:

(1) (Verify $T_i \cup T_{i+1}$ are feasible.) Based on the claim that we just confirmed we know that if any one of the inequalities

$$T_i \cup T_{i+1} \neq \emptyset$$

fails, then the reconstruction fails and no preimage can exist.
(2) (Reconstruct the set of deleted elements.) Put $e_1 = \max(T_1 \cup T_2)$, $e_2 = \min(T_2 \cup T_3), \ldots$.
(3) (Recover the choice sequence **h**.) For each $i = 1, \ldots, s - 1$, since $e_i \in T_i \cup T_{i+1}$, and because the T_i's are pairwise disjoint, there will be exactly one index, h_i, say, such that $h_i \in \{i, i+1\}$ and $e_i \notin T_{h_i}$.
(4) (Re-insert the elements that were deleted.) For each i, $1 \leqslant i < s$, insert the element e_i into the set T_{h_i}.

If the reconstructed sets $(S_1, \ldots, S_s)$ satisfy (12)–(14) then we have found the unique preimage. Otherwise no preimage exists.   □

Thus if $\hat{P}(n, s)$ is the number of permutations of n letters with s runs, the first of which is up, then we have shown that

$$2^{s-1} \hat{P}(n, s) \leqslant s^n. \tag{15}$$

5.5. When does the algorithm terminate without a preimage?

If the reconstruction algorithm does not early exit in step (1), yet fails to find a preimage, then one of the conditions (12)–(14) is not satisfied. We will now visit each of these in turn to see when it might fail. We will separate condition (12) into two subconditions: condition (12a) asserts that $|S_i \cap S_{i+1}| = 1$; and condition (12b) asserts that $|S_i \cap S_j| = 0$ for $j \geqslant i + 2$.

(1) (Can (12a) fail?) No. The intersections $T_i \cap T_{i+1}$ were all empty before the insertions; however, the operation, "insert element e_i into T_{h_i}" either added an element of T_i to the set T_{i+1}, or vice-versa. That operation alone caused the two adjacent sets to have intersection 1. The only other insertion which could have affected T_i is the one which involves element e_{i-1}. If that operation increased the size of T_i, then it did so by inserting an element from T_{i-1}, which element could not possibly be present in T_{i+1}. Thus, the only other insertion which could possibly affect the set T_i will have no effect on the cardinality of $T_i \cap T_{i+1}$. Likewise, the only other operation which can possibly affect the cardinality of T_{i+1} will have no effect on the cardinality of $T_i \cap T_{i+1}$. So, the intersection $S_i \cap S_{i+1}$ will always have size 1, as required.

(2) (Can (12b) fail?) Yes. Only the case $j = i + 2$ requires attention. If $S_i \cap S_{i+2}$ is not empty, then during reconstruction some element originally belonging to T_{i+1} was inserted into both T_i and T_{i+2}. (Any element originally in T_i cannot end up in S_{i+2}, and vice-versa.) This means that some element $e \in T_{i+1}$ is both the maximum of $T_i \cup T_{i+1}$, as well as the minimum of $T_{i+1} \cup T_{i+2}$ (or the other way around). But

$$\max(T_{i+1}) \leqslant \max(T_i \cup T_{i+1}) = e_i = \min(T_{i+1} \cup T_{i+2}) \leqslant \min(T_{i+1});$$

so, if condition (12b) fails for the reconstructed S_i's, then there must be a set T_{i+1} with just one element.

(3) (Can (13) fail?) Yes, if one of the sets T_i has cardinality 0 or 1, then it is possible that not enough elements will be inserted into T_i to bring its cardinality up to 2.

(4) (Can (14) fail?) No. By the nature of the reconstruction, the S_i's always have this property.

We can now prove

Lemma 5. *If in the given sequence $T = (T_1, \ldots, T_s)$, all sets have cardinalities at least 2, then T has a preimage under Φ.*

For then the unions $T_i \cup T_{i+1}$ have size 4 or more, so we do not terminate the reconstruction at step (1). The only other two possible failures—when an intersection $S_i \cap S_{i+2}$ was nonempty, or one of the S_i was too small—were both traced back in the above analysis to a set T_i which had size 0 or 1.

A crude lower estimate from Bonferroni's inequalities tells us that the number of s-tuples T that are pairwise disjoint, with union equal to $[n]$, and with all cardinalities $\geqslant 2$ is at least

$$s^n - (n + s)(s - 1)^{n-1}.$$

The reason: $s(s - 1)^n$ is an upper bound on the number of T's for which some component is the empty set; and $ns(s - 1)^{n-1}$ is an upper bound on the number of T's for which some component has cardinality one; then, $s(s - 1) + ns < s(n + s)$. Hence

$$2^{s-1} \hat{P}(n, s) \geqslant s^n - s(n + s)(s - 1)^{n-1}, \tag{16}$$

which, taken together with (15) completes the proof of (11), since our hypothesis on the pairs (n, s) implies

$$(n + s)(s - 1)^{n-1} \leqslant 2n(s - 1)^{n-1} = o(s^n).$$

5.6. The asymptotic series

To justify our claim that

$$P(n, s) \sim \psi_0(n, s)s^n + \psi_1(n, s)(s - 1)^n + \cdots$$

is an asymptotic series uniformly for $s \leqslant (1 - \epsilon)n/\log n$, it is necessary to know that each term in the above sum is little-oh of its predecessor, uniformly for s in the stated range; that is,

$$\frac{\psi_{i+1}(n, s)(s - i - 1)^n}{\psi_i(n, s)(s - i)^n} \to 0.$$

One is tempted to say we have the quotient of two polynomials in n, whose degrees differ by at most 1. Thus, the ratio $\psi_{i+1}(n, s)/\psi_i(n, s)$ is $O(n)$, and the assertion follows. However, this ignores the issue of how the coefficients of these polynomials depend on the parameter s. It is necessary to observe, using the recursion (8) and a simple induction, plus the known formula for $K(s)$, that $\psi_i(n, s)$ is 2^{-s-i+2} times a polynomial in n and s whose coefficients are rational and whose total degree is $\lfloor i/2 \rfloor$. With this observation in place, the earlier argument concerning $\psi_{i+1}(n, s)/\psi_i(n, s)$ is valid, uniformly for s in the stated range.

6. Epilogue

Although the exact formula for $P(n, s)$ presented here is not the first to appear, we would like to think our derivation and observations offer some novel aspects which we summarize here. History shows the generating function for these numbers can be elusive. Our "column" approach provides immediate access to a formula complete except for the parameter $K(s)$. We know, however, that $K(s)$ is the limit on n of the ratio $P(n, s)/s^n$, and the latter limit is determined in a separate argument. The limit is obtained by proving an asymptotic formula for $P(n, s)$ using a combinatorial argument which is an almost-bijection. Finally, it is observed that the asymptotic formula holds in a wider range of s, and in this wider range the exact formula for $P(n, s)$ has the additional feature of being a complete asymptotic series.

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