

# Total Positivity: Tests and Parametrizations

## Introduction

A matrix is *totally positive* (resp. *totally non-negative*) if all its minors are positive (resp. non-negative) real numbers. The first systematic study of these classes of matrices was undertaken in the 1930s by F. R. Gantmacher and M. G. Krein [20–22], who established their remarkable spectral properties (in particular, an  $n \times n$  totally positive matrix  $x$  has  $n$  distinct positive eigenvalues). Earlier, I. J. Schoenberg [41] had discovered the connection between total non-negativity and the following *variation-diminishing property*: the number of sign changes in a vector does not increase upon multiplying by  $x$ .

Total positivity found numerous applications and was studied from many different angles. An incomplete list includes oscillations in mechanical systems (the original motivation in [22]), stochastic processes and approximation theory [25, 28], Pólya frequency sequences [28, 40], representation theory of the infinite symmetric group and the Edrei–Thoma theorem [13, 44], planar resistor networks [11], unimodality and log-concavity [42], and theory of immanants [43]. Further references can be found in S. Karlin’s book [28] and in the surveys [2, 5, 38].

In this article, we focus on the following two problems:

1. *parametrizing* all totally non-negative matrices
2. *testing* a matrix for total positivity

Our interest in these problems stemmed from a surprising representation-theoretic connection between total

positivity and canonical bases for quantum groups, discovered by G. Lusztig [33] (cf. also the surveys in [31, 34]). Among other things, he extended the subject by defining totally positive and totally non-negative elements for any reductive group. Further development of these ideas in [3, 4, 15, 17] aims at generalizing the whole body of classical determinantal calculus to any semisimple group.

As often happens, putting things in a more general perspective shed new light on this classical subject. In the next two sections, we provide self-contained proofs (many of them new) of the fundamental results on problems 1 and 2, due to A. Whitney [46], C. Loewner [32], C. Cryer [9, 10], and M. Gasca and J. M. Peña [23]. The rest of the article presents more recent results obtained in [15]: a family of efficient total positivity criteria and explicit formulas for expanding a generic matrix into a product of elementary Jacobi matrices. These results and their proofs can be generalized to arbitrary semisimple groups [4, 15], but we do not discuss this here.

Our approach to the subject relies on two combinatorial constructions. The first one is well known: it associates a totally non-negative matrix to a planar directed graph with positively weighted edges (in fact, every totally non-negative matrix can be obtained in this way [6]). Our second combinatorial tool was introduced in [15]; it is a particular class of colored pseudoline arrangements that we call the *double wiring diagrams*.

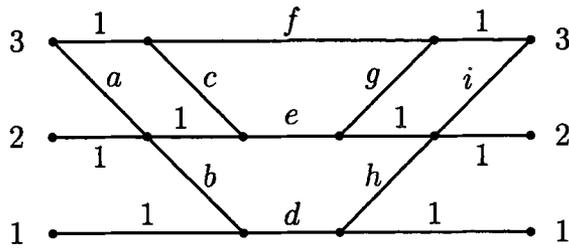


Figure 1. A planar network.

### Planar Networks

To the uninitiated, it might be unclear that totally positive matrices of arbitrary order exist at all. As a warm-up, we invite the reader to check that every matrix given by

$$\begin{bmatrix} d & dh & dhi \\ bd & bdh + e & bdhi + eg + ei \\ abd & abdh + ae + ce & abdhi + (a + c)e(g + i) + f \end{bmatrix}, \quad (1)$$

where the numbers  $a, b, c, d, e, f, g, h, i$  are positive, is totally positive. It will follow from the results later that every  $3 \times 3$  totally positive matrix has this form.

We will now describe a general procedure that produces totally non-negative matrices. In what follows, a *planar network*  $(\Gamma, \omega)$  is an acyclic directed planar graph  $\Gamma$  whose edges  $e$  are assigned scalar *weights*  $\omega(e)$ . In all of our examples (cf. Figures 1, 2, 5), we assume the edges of  $\Gamma$  directed left to right. Also, each of our networks will have  $n$  sources and  $n$  sinks, located at the left (resp. right) edge of the picture, and numbered bottom to top.

The *weight* of a directed path in  $\Gamma$  is defined as the product of the weights of its edges. The *weight matrix*  $x(\Gamma, \omega)$  is an  $n \times n$  matrix whose  $(i, j)$ -entry is the sum of weights of all paths from the source  $i$  to the sink  $j$ ; for example, the weight matrix of the network in Figure 1 is given by (1).

The minors of the weight matrix of a planar network have an important combinatorial interpretation, which can be traced to B. Lindström [30] and further to S. Karlin and G. McGregor [29] (implicit), and whose many applications were given by I. Gessel and G. X. Viennot [26, 27].

In what follows,  $\Delta_{I,J}(x)$  denotes the minor of a matrix  $x$  with the row set  $I$  and the column set  $J$ .

The weight of a collection of directed paths in  $\Gamma$  is defined to be the product of their weights.

**LEMMA 1** (Lindström’s Lemma). *A minor  $\Delta_{I,J}$  of the weight matrix of a planar network is equal to the sum of weights of all collections of vertex-disjoint paths that connect the sources labeled by  $I$  with the sinks labeled by  $J$ .*

To illustrate, consider the matrix  $x$  in (1). We have, for example,  $\Delta_{23,23}(x) = bcdegh + bdfh + fe$ , which also equals the sum of the weights of the three vertex-disjoint path collections in Figure 1 that connect sources 2 and 3 to sinks 2 and 3.

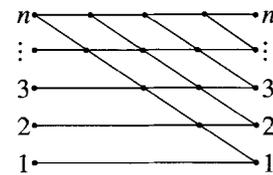
*Proof.* It suffices to prove the lemma for the determinant of the whole weight matrix  $x = x(\Gamma, \omega)$  (i.e., for the case  $I = J = [1, n]$ ). Expanding the determinant, we obtain

$$\det(x) = \sum_w \sum_{\pi} \operatorname{sgn}(\omega) \omega(\pi), \quad (2)$$

the sum being over all permutations  $w$  in the symmetric group  $S_n$  and over all collections of paths  $\pi = (\pi_1, \dots, \pi_n)$  such that  $\pi_i$  joins the source  $i$  with the sink  $w(i)$ . Any collection  $\pi$  of vertex-disjoint paths is associated with the identity permutation; hence,  $\omega(\pi)$  appears in (2) with the positive sign. We need to show that all other terms in (2) cancel out. Deforming  $\Gamma$  a bit if necessary, we may assume that no two vertices lie on the same vertical line. This makes the following involution on the non-vertex-disjoint collections of paths well defined: take the leftmost common vertex of two or more paths in  $\pi$ , take two smallest indices  $i$  and  $j$  such that  $\pi_i$  and  $\pi_j$  contain  $v$ , and switch the parts of  $\pi_i$  and  $\pi_j$  lying to the left of  $v$ . This involution preserves the weight of  $\pi$  while changing the sign of the associated permutation  $w$ ; the corresponding pairing of terms in (2) provides the desired cancellation.  $\square$

**COROLLARY 2.** *If a planar network has non-negative real weights, then its weight matrix is totally non-negative.*

As an aside, note that the weight matrix of the network



(with unit edge weights) is the “Pascal triangle”

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & 0 & \dots \\ 1 & 4 & 6 & 4 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

which is totally non-negative by Corollary 2. Similar arguments can be used to show total non-negativity of various other combinatorial matrices, such as the matrices of  $q$ -binomial coefficients, Stirling numbers of both kinds, and so forth.

We call a planar network  $\Gamma$  *totally connected* if for any two subsets  $I, J \subset [1, n]$  of the same cardinality, there exists a collection of vertex-disjoint paths in  $\Gamma$  connecting the sources labeled by  $I$  with the sinks labeled by  $J$ .

**COROLLARY 3.** *If a totally connected planar network has positive weights, then its weight matrix is totally positive.*

For any  $n$ , let  $\Gamma_0$  denote the network shown in Figure 2. Direct inspection shows that  $\Gamma_0$  is totally connected.

**COROLLARY 4.** *For any choice of positive weights  $\omega(e)$ , the weight matrix  $x(\Gamma_0, \omega)$  is totally positive.*

It turns out that this construction produces *all* totally positive matrices; this result is essentially equivalent to

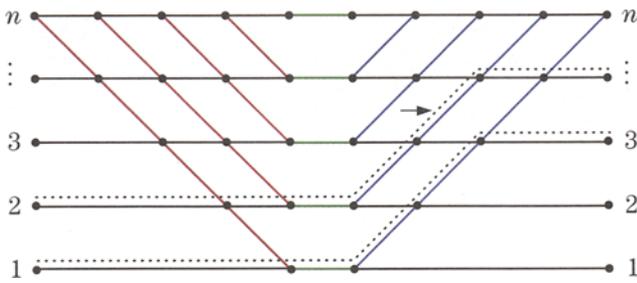


Figure 2. Planar network  $\Gamma_0$ .

A. Whitney's Reduction Theorem [46] and can be sharpened as follows. Call an edge of  $\Gamma_0$  *essential* if it either is slanted or is one of the  $n$  horizontal edges in the middle of the network. Note that  $\Gamma_0$  has exactly  $n^2$  essential edges. A weighting  $\omega$  of  $\Gamma_0$  is *essential* if  $\omega(e) \neq 0$  for any essential edge  $e$  and  $\omega(e) = 1$  for all other edges.

**THEOREM 5.** *The map  $\omega \mapsto x(\Gamma_0, \omega)$  restricts to a bijection between the set of all essential positive weightings of  $\Gamma_0$  and the set of all totally positive  $n \times n$  matrices.*

The proof of this theorem will use the following notions. A minor  $\Delta_{I,J}$  is called *solid* if both  $I$  and  $J$  consist of several consecutive indices; if, furthermore,  $I \cup J$  contains 1, then  $\Delta_{I,J}$  is called *initial* (see Fig. 3). Each matrix entry is the lower-right corner of exactly one initial minor; thus, the total number of such minors is  $n^2$ .

**LEMMA 6.** *The  $n^2$  weights of essential edges in an essential weighting  $\omega$  of  $\Gamma_0$  are related to the  $n^2$  initial minors of the weight matrix  $x = x(\Gamma_0, \omega)$  by an invertible monomial transformation. Thus, an essential weighting  $\omega$  of  $\Gamma_0$  is uniquely recovered from  $x$ .*

*Proof.* The network  $\Gamma_0$  has the following easily verified property: For any set  $I$  of  $k$  consecutive indices in  $[1, n]$ , there is a unique collection of  $k$  vertex-disjoint paths connecting the sources labeled by  $[1, k]$  (resp. by  $I$ ) with the sinks labeled by  $I$  (resp. by  $[1, k]$ ). These paths are shown by dotted lines in Figure 2, for  $k = 2$  and  $I = [3, 4]$ . By Lindström's lemma, every initial minor  $\Delta$  of  $x(\Gamma_0, \omega)$  is equal to the product of the weights of essential edges covered by this family of paths. Note that among these edges, there is always a unique *uppermost* essential edge  $e(\Delta)$  (indicated by the arrow in Figure 2). Furthermore, the map  $\Delta \mapsto e(\Delta)$  is a bijection between initial minors and essential edges. It follows that the weight of each essential edge  $e = e(\Delta)$  is equal to  $\Delta$  times a Laurent monomial in some initial minors  $\Delta'$ , whose associated edges  $e(\Delta')$  are located below  $e$ .  $\square$

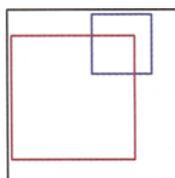


Figure 3. Initial minors.

To illustrate Lemma 6, consider the special case  $n = 3$ . The network  $\Gamma_0$  is shown in Figure 1; its essential edges have the weights  $a, b, \dots, i$ . The weight matrix  $x(\Gamma_0, \omega)$  is given in (1). Its initial minors are given by the monomials

$$\begin{aligned} \Delta_{1,1} &= \underline{a}, & \Delta_{1,2} &= \underline{dh}, & \Delta_{1,3} &= \underline{dhi}, \\ \Delta_{2,1} &= \underline{bd}, & \Delta_{12,12} &= \underline{de}, & \Delta_{12,23} &= \underline{degh}, \\ \Delta_{3,1} &= \underline{abd}, & \Delta_{23,12} &= \underline{bcde}, & \Delta_{123,123} &= \underline{def}, \end{aligned}$$

where for each minor  $\Delta$ , the "leading entry"  $\omega(e(\Delta))$  is underlined.

To complete the proof of Theorem 5, it remains to show that every totally positive matrix  $x$  has the form  $x(\Gamma_0, \omega)$  for some essential positive weighting  $\omega$ . By Lemma 6, such an  $\omega$  can be chosen so that  $x$  and  $x(\Gamma_0, \omega)$  will have the same initial minors. Thus, our claim will follow from Lemma 7.

**LEMMA 7.** *A square matrix  $x$  is uniquely determined by its initial minors, provided all these minors are nonzero.*

*Proof.* Let us show that each matrix entry  $x_{ij}$  of  $x$  is uniquely determined by the initial minors. If  $i = 1$  or  $j = 1$ , there is nothing to prove, since  $x_{ij}$  is itself an initial minor. Assume that  $\min(i, j) > 1$ . Let  $\Delta$  be the initial minor whose last row is  $i$  and last column is  $j$ , and let  $\Delta'$  be the initial minor obtained from  $\Delta$  by deleting this row and this column. Then,  $\Delta = \Delta' x_{ij} + P$ , where  $P$  is a polynomial in the matrix entries  $x_{i',j'}$  with  $(i', j') \neq (i, j)$  and  $i' \leq i$  and  $j' \leq j$ . Using induction on  $i + j$ , we can assume that each  $x_{i',j'}$  that occurs in  $P$  is uniquely determined by the initial minors, so the same is true for  $x_{ij} = (\Delta - P)/\Delta'$ . This completes the proofs of Lemma 7 and Theorem 5.  $\square$

Theorem 5 describes a *parametrization* of totally positive matrices by  $n^2$ -tuples of positive reals, providing a partial answer (one of the many possible, as we will see) to the first problem stated in the Introduction. The second problem—that of *testing* total positivity of a matrix—can also be solved using this theorem, as we will now explain.

An  $n \times n$  matrix has altogether  $\binom{2n}{n} - 1$  minors. This makes it impractical to test positivity of every single minor. It is desirable to find efficient criteria for total positivity that would only check a small fraction of all minors.

**EXAMPLE 8.** A  $2 \times 2$  matrix

$$x = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

has  $\binom{4}{2} - 1 = 5$  minors: four matrix entries and the determinant  $\Delta = ad - bc$ . To test that  $x$  is totally positive, it is enough to check the positivity of  $a, b, c$ , and  $\Delta$ ; then,  $d = (\Delta + bc)/a > 0$ .

The following theorem generalizes this example to matrices of arbitrary size; it is a direct corollary of Theorem 5 and Lemmas 6 and 7.

**THEOREM 9.** *A square matrix is totally positive if and only if all its initial minors (see Fig. 3) are positive.*

This criterion involves  $n^2$  minors, and it can be shown that this number cannot be lessened. Theorem 9 was proved by M. Gasca and Peña [23, Theorem 4.1] (for rec-

tangular matrices); it also follows from Cryer's results in [9]. Theorem 9 is an enhancement of the 1912 criterion by M. Fekete [14], who proved that the positivity of all solid minors of a matrix implies its total positivity.

### Theorems of Whitney and Loewner

In this article, we shall only consider *invertible* totally non-negative  $n \times n$  matrices. Although these matrices have real entries, it is convenient to view them as elements of the general linear group  $G = GL_n(\mathbb{C})$ . We denote by  $G_{\geq 0}$  (resp.  $G_{> 0}$ ) the set of all totally non-negative (resp. totally positive) matrices in  $G$ . The structural theory of these matrices begins with the following basic observation, which is an immediate corollary of the Binet–Cauchy formula.

**PROPOSITION 10.** *Both  $G_{\geq 0}$  and  $G_{> 0}$  are closed under matrix multiplication. Furthermore, if  $x \in G_{\geq 0}$  and  $y \in G_{> 0}$ , then both  $xy$  and  $yx$  belong to  $G_{> 0}$ .*

Combining this proposition with the foregoing results, we will prove the following theorem of Whitney [46].

**THEOREM 11.** (Whitney's theorem). *Every invertible totally non-negative matrix is the limit of a sequence of totally positive matrices.*

Thus,  $G_{\geq 0}$  is the closure of  $G_{> 0}$  in  $G$ . (The condition of invertibility in Theorem 11 can, in fact, be lifted.)

*Proof.* First, let us show that the identity matrix  $I$  lies in the closure of  $G_{> 0}$ . By Corollary 4, it suffices to show that  $I = \lim_{N \rightarrow \infty} x(\Gamma_0, \omega_N)$  for some sequence of positive weightings  $\omega_N$  of the network  $\Gamma_0$ . Note that the map  $\omega \mapsto x(\Gamma_0, \omega)$  is continuous and choose any sequence of positive weightings that converges to the weighting  $\omega_0$  defined by  $\omega_0(e) = 1$  (resp. 0) for all horizontal (resp. slanted) edges  $e$ . Clearly,  $x(\Gamma_0, \omega_0) = I$ , as desired.

To complete the proof, write any matrix  $x \in G_{\geq 0}$  as  $x = \lim_{N \rightarrow \infty} x \cdot x(\Gamma_0, \omega_N)$ , and note that all matrices  $x \cdot x(\Gamma_0, \omega_N)$  are totally positive by Proposition 10.  $\square$

The following description of the multiplicative monoid  $G_{\geq 0}$  was first given by Loewner [32] under the name “Whitney's Theorem”; it can indeed be deduced from [46].

**THEOREM 12** (Loewner–Whitney theorem). *Any invertible totally non-negative matrix is a product of elementary Jacobi matrices with non-negative matrix entries.*

Here, an “elementary Jacobi matrix” is a matrix  $x \in G$  that differs from  $I$  in a single entry located either on the main diagonal or immediately above or below it.

*Proof.* We start with an inventory of elementary Jacobi matrices. Let  $E_{i,j}$  denote the  $n \times n$  matrix whose  $(i, j)$ -entry is 1 and all other entries are 0. For  $t \in \mathbb{C}$  and  $i = 1, \dots, n - 1$ , let

$$x_i(t) = I + tE_{i,i+1} = \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & t & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}$$

and

$$x_{\bar{i}}(t) = I + tE_{i+1,i} = (x_i(t))^T$$

(the transpose of  $x_i(t)$ ). Also, for  $i = 1, \dots, n$  and  $t \neq 0$ , let

$$x_{\textcircled{i}}(t) = I + (t - 1)E_{i,i},$$

the diagonal matrix with the  $i$ th diagonal entry equal to  $t$  and all other diagonal entries equal to 1. Thus, elementary Jacobi matrices are precisely the matrices of the form  $x_i(t)$ ,  $x_{\bar{i}}(t)$ , and  $x_{\textcircled{i}}(t)$ . An easy check shows that they are totally non-negative for any  $t > 0$ .

For any word  $\mathbf{i} = (i_1, \dots, i_l)$  in the alphabet

$$\mathcal{A} = \{1, \dots, n - 1, \textcircled{1}, \dots, \textcircled{n}, \bar{1}, \dots, \overline{n - 1}\}, \quad (3)$$

we define the *product map*  $x_{\mathbf{i}}: (\mathbb{C} \setminus \{0\})^l \rightarrow G$  by

$$x_{\mathbf{i}}(t_1, \dots, t_l) = x_{i_1}(t_1) \cdots x_{i_l}(t_l). \quad (4)$$

(Actually,  $x_{\mathbf{i}}(t_1, \dots, t_l)$  is well defined as long as the right-hand side of (4) does not involve any factors of the form  $x_{\textcircled{i}}(0)$ .) To illustrate, the word  $\mathbf{i} = \textcircled{1} \bar{1} \textcircled{2} 1$  gives rise to

$$x_{\mathbf{i}}(t_1, t_2, t_3, t_4) = \begin{bmatrix} t_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & t_3 \end{bmatrix} \begin{bmatrix} 1 & t_4 \\ 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} t_1 & t_1 t_4 \\ t_2 & t_2 t_4 + t_3 \end{bmatrix}.$$

We will interpret each matrix  $x_{\mathbf{i}}(t_1, \dots, t_l)$  as the weight matrix of a planar network. First, note that any elementary Jacobi matrix is the weight matrix of a “chip” of one of the three kinds shown in Figure 4. In each “chip,” all edges but one have weight 1; the distinguished edge has weight  $t$ . Slanted edges connect horizontal levels  $i$  and  $i + 1$ , counting from the bottom; in all examples in Figure 4,  $i = 2$ .

The weighted planar network  $(\Gamma(\mathbf{i}), \omega(t_1, \dots, t_l))$  is then constructed by concatenating the “chips” corresponding to consecutive factors  $x_{i_k}(t_k)$ , as shown in Figure 5. It is easy to see that concatenation of planar networks corresponds to multiplying their weight matrices. We conclude that the product  $x_{\mathbf{i}}(t_1, \dots, t_l)$  of elementary Jacobi matrices equals the weight matrix  $x(\Gamma(\mathbf{i}), \omega(t_1, \dots, t_l))$ .

In particular, the network  $(\Gamma_0, \omega)$  appearing in Figure 2 and Theorem 5 (more precisely, its equivalent deformation) corresponds to some special word  $\mathbf{i}_{\max}$  of length  $n^2$ ; instead of defining  $\mathbf{i}_{\max}$  formally, we just write it for  $n = 4$ :

$$\mathbf{i}_{\max} = (\bar{3}, \bar{2}, \bar{3}, \bar{1}, \bar{2}, \bar{3}, \textcircled{1}, \textcircled{2}, \textcircled{3}, \textcircled{4}, 3, 2, 3, 1, 2, 3).$$

In view of this, Theorem 5 can be reformulated as follows.

**THEOREM 13.** *The product map  $x_{\mathbf{i}_{\max}}$  restricts to a bijection between  $n^2$ -tuples of positive real numbers and totally positive  $n \times n$  matrices.*

We will prove the following refinement of Theorem 12, which is a reformulation of its original version [32].

**THEOREM 14.** *Every matrix  $x \in G_{\geq 0}$  can be written as  $x = x_{\mathbf{i}_{\max}}(t_1, \dots, t_{n^2})$ , for some  $t_1, \dots, t_{n^2} \geq 0$ .*

(Since  $x$  is invertible, we must in fact have  $t_k > 0$  for  $n(n-1)/2 < k \leq n(n+1)/2$  (i.e., for those indices  $k$  for which the corresponding entry of  $\mathbf{i}_{\max}$  is of the form  $(i)$ ).

*Proof.* The following key lemma is due to Cryer [9].

**LEMMA 15.** *The leading principal minors  $\Delta_{[1,k],[1,k]}$  of a matrix  $x \in G_{\geq 0}$  are positive for  $k = 1, \dots, n$ .*

*Proof.* Using induction on  $n$ , it suffices to show that  $\Delta_{[1,n-1],[1,n-1]}(x) > 0$ . Let  $\Delta^{i,j}(x)$  [resp.  $\Delta^{i',j'}(x)$ ] denote the minor of  $x$  obtained by deleting the row  $i$  and the column  $j$  (resp. rows  $i$  and  $i'$ , and columns  $j$  and  $j'$ ). Then, for any  $1 \leq i < i' \leq n$  and  $1 \leq j < j' \leq n$ , one has

$$\Delta^{i',j'}(x) \Delta^{i,j}(x) - \Delta^{i',j}(x) \Delta^{i,j'}(x) = \det(x) \Delta^{i',j'}(x) \quad (5)$$

as an immediate consequence of Jacobi's formula for minors of the inverse matrix (see, e.g., [7, Lemma 9.2.10]). The determinantal identity (5) was proved by Desnanot as early as in 1819 (see [37, pp. 140–142]); it is sometimes called “Lewis Carroll's identity,” due to the role it plays in C. L. Dodgson's condensation method [12, pp. 170–180].

Now suppose that  $\Delta^{n,n}(x) = 0$  for some  $x \in G_{\geq 0}$ . Because  $x$  is invertible, we have  $\Delta^{i,n}(x) > 0$  and  $\Delta^{n,j}(x) > 0$  for some indices  $i, j < n$ . Using (5) with  $i' = j' = n$ , we arrive at a desired contradiction by

$$0 > -\Delta^{n,j}(x) \Delta^{i,n}(x) = \det(x) \Delta^{i,n,jn}(x) \geq 0. \quad \square$$

We are now ready to complete the proof of Theorem 14. Any matrix  $x \in G_{\geq 0}$  is by Theorem 11 a limit of totally positive matrices  $x_N$ , each of which can, by Theorem 13, be factored as  $x_N = x_{\mathbf{i}_{\max}}(t_1^{(N)}, \dots, t_{n^2}^{(N)})$  with all  $t_k^{(N)}$  positive. It suffices to show that the sequence  $s_N = \sum_{k=1}^{n^2} t_k^{(N)}$  converges; then, the standard compactness argument will imply that the sequence of vectors  $(t_1^{(N)}, \dots, t_{n^2}^{(N)})$  contains a converging subsequence, whose limit  $(t_1, \dots, t_{n^2})$  will provide the desired factorization  $x = x_{\mathbf{i}_{\max}}(t_1, \dots, t_{n^2})$ . To see that  $(s_N)$  converges, we use the explicit formula

$$s_N = \sum_{i=1}^n \frac{\Delta_{[1,i],[1,i]}(x_N)}{\Delta_{[1,i-1],[1,i-1]}(x_N)} + \sum_{i=1}^{n-1} \frac{\Delta_{[1,i-1] \cup \{i+1\}, [1,i]}(x_N) + \Delta_{[1,i],[1,i-1] \cup \{i+1\}}(x_N)}{\Delta_{[1,i],[1,i]}(x_N)}$$

(to prove this, compute the minors on the right with the help of Lindström's lemma and simplify). Thus,  $s_N$  is expressed as a Laurent polynomial in the minors of  $x_N$  whose denominators only involve leading principal minors  $\Delta_{[1,k],[1,k]}$ . By Lemma 15, as  $x_N$  converges to  $x$ , this Laurent polynomial converges to its value at  $x$ . This completes the proofs of Theorems 12 and 14.  $\square$

### Double Wiring Diagrams and Total Positivity Criteria

We will now give another proof of Theorem 9, which will include it into a family of “optimal” total positivity criteria that correspond to combinatorial objects called *double wiring diagrams*. This notion is best explained by an example, such as the one given in Figure 6. A double wiring

diagram consists of two families of  $n$  piecewise-straight lines (each family colored with one of the two colors), the crucial requirement being that each pair of lines of like color intersect exactly once.

The lines in a double wiring diagram are numbered separately within each color. We then assign to every chamber of a diagram a pair of subsets of the set  $[1, n] = \{1, \dots, n\}$ ; each subset indicates which lines of the corresponding color pass below that chamber; see Figure 7.

Thus, every chamber is naturally associated with a minor  $\Delta_{I,J}$  of an  $n \times n$  matrix  $x = (x_{ij})$  (we call it a *chamber minor*) that occupies the rows and columns specified by the sets  $I$  and  $J$  written inside that chamber. In our running example, there are nine chamber minors (the total number is always  $n^2$ ), namely  $x_{31}, x_{32}, x_{12}, x_{13}, \Delta_{23,12}, \Delta_{13,12}, \Delta_{13,23}, \Delta_{12,23}$ , and  $\Delta_{123,123} = \det(x)$ .

**THEOREM 16.** *Every double wiring diagram gives rise to the following criterion: an  $n \times n$  matrix is totally positive if and only if all its  $n^2$  chamber minors are positive.*

The criterion in Theorem 9 is a special case of Theorem 16 and arises from the “lexicographically minimal” double wiring diagram, shown in Figure 8 for  $n = 3$ .

*Proof.* We will actually prove the following statement that implies Theorem 16.

**THEOREM 17.** *Every minor of a generic square matrix can be written as a rational expression in the chamber minors of a given double wiring diagram, and, moreover, this rational expression is subtraction-free (i.e., all coefficients in the numerator and denominator are positive).*

Two double wiring diagrams are called *isotopic* if they have the same collections of chamber minors. The terminology suggests what is really going on here: two isotopic diagrams have the same “topology.” From now on, we will treat such diagrams as indistinguishable from each other.

We will deduce Theorem 17 from the following fact: any two double wiring diagrams can be transformed into each other by a sequence of local “moves” of three different kinds, shown in Figure 9. (This is a direct corollary of a theorem of G. Ringel [39]. It can also be derived from the Tits theorem on reduced words in the symmetric group; cf. (7) and (8) below.)

Note that each local move exchanges a single chamber minor  $Y$  with another chamber minor  $Z$  and keeps all other chamber minors in place.

**LEMMA 18.** *Whenever two double wiring diagrams differ by a single local move of one of the three types shown in Figure 9, the chamber minors appearing there satisfy the identity  $AC + BD = YZ$ .*

The three-term determinantal identities of Lemma 18 are well known, although not in this disguised form. The last of these identities is nothing but the identity (5), applied to various submatrices of an  $n \times n$  matrix. The identities corresponding to the top two “moves” in Figure 9 are special instances of the classical Grassmann–Plücker relations (see,

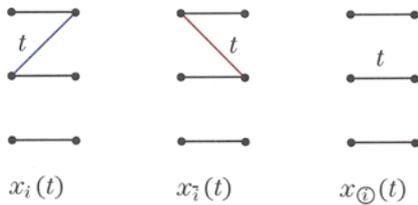


Figure 4. Elementary “chips.”

e.g., [18, (15.53)]), and were obtained by Desnanot alongside (5) in the same 1819 publication we mentioned earlier.

Theorem 17 is now proved as follows. We first note that any minor appears as a chamber minor in some double wiring diagram. Therefore, it suffices to show that the chamber minors of one diagram can be written as subtraction-free rational expressions in the chamber minors of any other diagram. This is a direct corollary of Lemma 18 combined with the fact that any two diagrams are related by a sequence of local moves: indeed, each local move replaces  $Y$  by  $(AC + BD)/Z$ , or  $Z$  by  $(AC + BD)/Y$ .  $\square$

Implicit in the above proof is an important combinatorial structure lying behind Theorems 16 and 17: the graph  $\Phi_n$ , whose vertices are the (isotopy classes of) double wiring diagrams and whose edges correspond to local moves. The study of  $\Phi_n$  is an interesting problem in itself. The first nontrivial example is the graph  $\Phi_3$  shown in Figure 10. It has 34 vertices, corresponding to 34 different total positivity criteria. Each of these criteria tests nine minors of a  $3 \times 3$  matrix. Five of these minors [viz.,  $x_{31}$ ,  $x_{13}$ ,  $\Delta_{23,12}$ ,  $\Delta_{12,23}$ , and  $\det(x)$ ] correspond to the “unbounded” chambers that lie on the periphery of every double wiring diagram; they are common to all 34 criteria. The other four minors correspond to the bounded chambers and depend on the choice of a diagram. For example, the criterion derived from Figure 7 involves “bounded” chamber minors  $\Delta_{3,2}$ ,  $\Delta_{1,2}$ ,  $\Delta_{13,12}$ , and  $\Delta_{13,23}$ . In Figure 10, each vertex of  $\Phi_3$  is labeled by the quadruple of “bounded” minors that appear in the corresponding total positivity criterion.

We suggest the following refinement of Theorem 17.

**CONJECTURE 19.** *Every minor of a generic square matrix can be written as a Laurent polynomial with non-negative integer coefficients in the chamber minors of an arbitrary double wiring diagram.*

Perhaps more important than proving this conjecture would be to give explicit combinatorial expressions for the

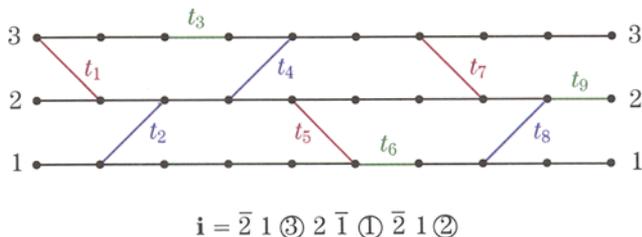


Figure 5. Planar network  $\Gamma(i)$ .

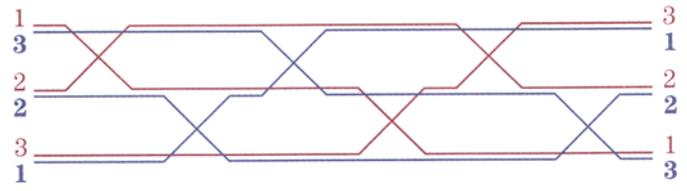


Figure 6. Double wiring diagram.

Laurent polynomials in question. We note a case in which the conjecture is true and the desired expressions can be given: the “lexicographically minimal” double wiring diagram whose chamber minors are the initial minors. Indeed, a generic matrix  $x$  can be uniquely written as the product  $x_{i_{\max}}(t_1, \dots, t_{n^2})$  of elementary Jacobi matrices (cf. Theorem 13); then, each minor of  $x$  can be written as a polynomial in the  $t_k$  with non-negative integer coefficients (with the help of Lindström’s lemma), whereas each  $t_k$  is a Laurent monomial in the initial minors of  $x$ , by Lemma 6.

It is proved in [15, Theorem 1.13] that every minor can be written as a Laurent polynomial with integer (possibly negative) coefficients in the chamber minors of a given diagram. Note, however, that this result combined with Theorem 17, does not imply Conjecture 19, because there do exist subtraction-free rational expressions that are Laurent polynomials, although *not* with non-negative coefficients (e.g., think of  $(p^3 + q^3)/(p + q) = p^2 - pq + q^2$ ).

The following special case of Conjecture 19 can be derived from [3, Theorem 3.7.4].

**THEOREM 20.** *Conjecture 19 holds for all wiring diagrams in which all intersections of one color precede the intersections of another color.*

We do not know an elementary proof of this result; the proof in [3] depends on the theory of canonical bases for quantum general linear groups.

#### Digression: Somos sequences

The three-term relation  $AC + BD = YZ$  is surrounded by some magic that eludes our comprehension. We cannot resist mentioning the related problem involving the *Somos-5 sequences* [19]. (We thank Richard Stanley for telling us about them.) These are the sequences  $a_1, a_2, \dots$  in which any six consecutive terms satisfy this relation:

$$a_n a_{n+5} = a_{n+1} a_{n+4} + a_{n+2} a_{n+3}. \quad (6)$$

Each term of a Somos-5 sequence is obviously a subtraction-free rational expression in the first five terms  $a_1, \dots, a_5$ . It can be shown by extending the arguments in [19, 35]

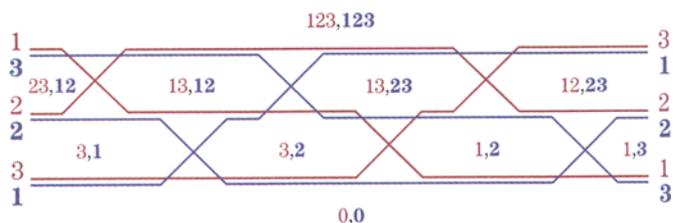


Figure 7. Chamber minors.

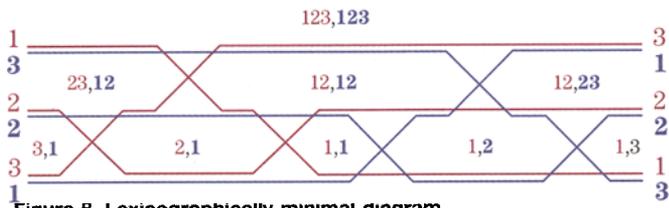


Figure 8. Lexicographically minimal diagram.

that each  $a_n$  is actually a Laurent polynomial in  $a_1, \dots, a_5$ . This property is truly remarkable, given the nature of the recurrence, and the fact that, as  $n$  grows, these Laurent polynomials become huge sums of monomials involving large coefficients; still, each of these sums cancels out from the denominator of the recurrence relation  $a_{n+5} = (a_{n+1}a_{n+4} + a_{n+2}a_{n+3})/a_n$ .

We suggest the following analog of Conjecture 19.

**CONJECTURE 21.** *Every term of a Somos-5 sequence is a Laurent polynomial with non-negative integer coefficients in the first five terms of the sequence.*

**Factorization Schemes**

According to Theorem 16, every double wiring diagram gives rise to an “optimal” total positivity criterion. We will now show that double wiring diagrams can be used to obtain a family of bijective parametrizations of the set  $G_{>0}$  of all totally positive matrices; this family will include the parametrization in Theorem 13 as a special case.

We encode a double wiring diagram by the word of length  $n(n - 1)$  in the alphabet  $\{1, \dots, n - 1, \bar{1}, \dots, \bar{n - 1}\}$  obtained by recording the heights of intersections of pseudolines of like color (traced left to right; barred digits for red crossings, unbarred for blue). For example, the diagram in Figure 6 is encoded by the word  $\bar{2} \bar{1} 2 \bar{1} \bar{2} 1$ .

The words that encode double wiring diagrams have an alternative description in terms of *reduced expressions* in the symmetric group  $S_n$ . Recall that by a famous theorem of E. H. Moore [36],  $S_n$  is a *Coxeter group* of type  $A_{n-1}$ ; that is, it is generated by the involutions  $s_1, \dots, s_{n-1}$  (adjacent transpositions) subject to the relations  $s_i s_j = s_j s_i$  for

$|i - j| \geq 2$ , and  $s_i s_j s_i = s_j s_i s_j$  for  $|i - j| = 1$ . A *reduced word* for a permutation  $w \in S_n$  is a word  $\mathbf{j} = (j_1, \dots, j_l)$  of the shortest possible length  $l = \ell(w)$  that satisfies  $w = s_{j_1} \cdots s_{j_l}$ . The number  $\ell(w)$  is called the *length* of  $w$  (it is the number of inversions in  $w$ ). The group  $S_n$  has a unique element  $w_0$  of maximal length: the order-reversing permutation of  $1, \dots, n$ ; it gives  $\ell(w_0) = \binom{n}{2}$ .

It is straightforward to verify that the encodings of double wiring diagrams are precisely the shuffles of two reduced words for  $w_0$ , in the barred and unbarred entries, respectively; equivalently, these are the reduced words for the element  $(w_0, w_0)$  of the Coxeter group  $S_n \times S_n$ .

**DEFINITION 22.** A word  $\mathbf{i}$  in the alphabet  $\mathcal{A}$  (see (3)) is called a *factorization scheme* if it contains each circled entry  $\textcircled{i}$  exactly once, and the remaining entries encode the heights of intersections in a double wiring diagram.

Equivalently, a factorization scheme  $\mathbf{i}$  is a shuffle of two reduced words for  $w_0$  (one barred and one unbarred) and an arbitrary permutation of the entries  $\textcircled{1}, \dots, \textcircled{n}$ . In particular,  $\mathbf{i}$  consists of  $n^2$  entries.

To illustrate, the word  $\mathbf{i} = \bar{2} 1 \textcircled{3} 2 \bar{1} \textcircled{1} \bar{2} 1 \textcircled{2}$ , appearing in Figure 5 is a factorization scheme.

An important example of a factorization scheme is the word  $\mathbf{i}_{\max}$  introduced in Theorem 13. Thus, the following result generalizes Theorem 13.

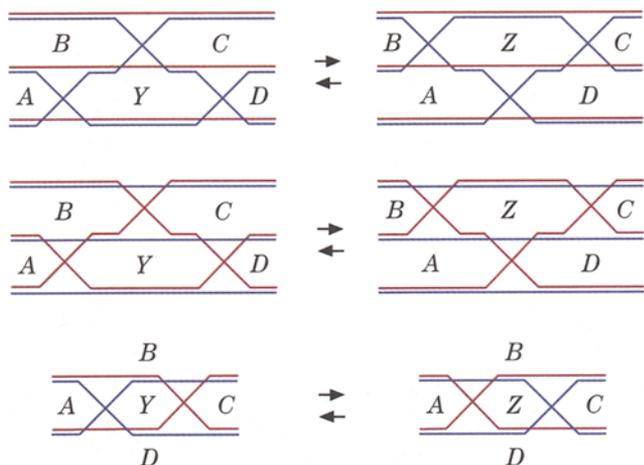
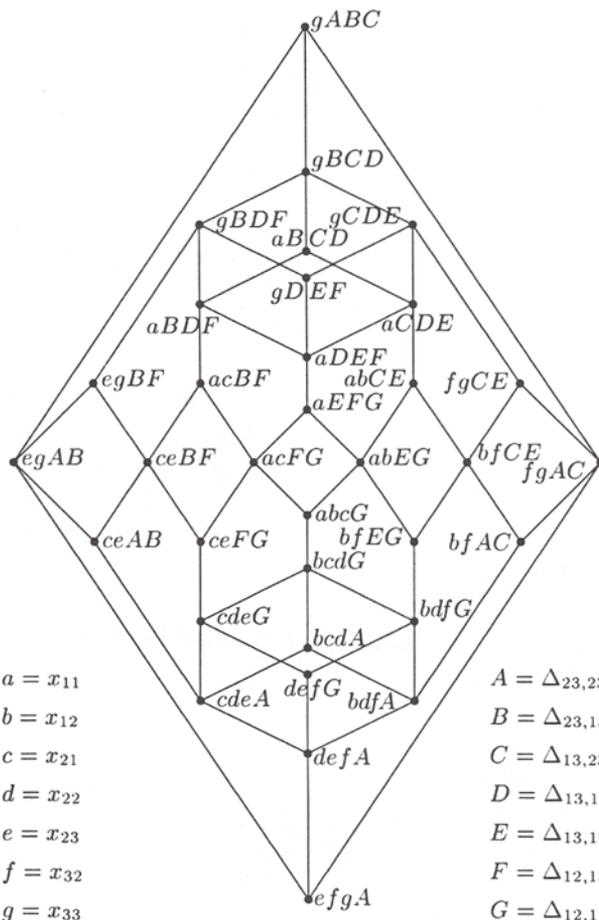


Figure 9. Local “moves.”



- $a = x_{11}$
- $b = x_{12}$
- $c = x_{21}$
- $d = x_{22}$
- $e = x_{23}$
- $f = x_{32}$
- $g = x_{33}$
- $A = \Delta_{23,23}$
- $B = \Delta_{23,13}$
- $C = \Delta_{13,23}$
- $D = \Delta_{13,13}$
- $E = \Delta_{13,12}$
- $F = \Delta_{12,13}$
- $G = \Delta_{12,12}$

Figure 10. Total positivity criteria for  $GL_3$ .

**THEOREM 23** [15]. *For an arbitrary factorization scheme  $\mathbf{i} = (i_1, \dots, i_{n^2})$ , the product map  $x_{\mathbf{i}}$  given by (4) restricts to a bijection between  $n^2$ -tuples of positive real numbers and totally positive  $n \times n$  matrices.*

*Proof.* We have already stated that any two double wiring diagrams are connected by a succession of the local “moves” shown in Figure 9. In the language of factorization schemes, this translates into any two factorization schemes being connected by a sequence of local transformations of the form

$$\begin{aligned} \cdots \bar{i} j i \cdots &\rightsquigarrow \cdots j i j \cdots, & |i - j| = 1, \\ \cdots \bar{i} j i \cdots &\rightsquigarrow \cdots \bar{j} i j \cdots, & |i - j| = 1, \end{aligned} \quad (7)$$

or of the form

$$\cdots ab \cdots \rightsquigarrow \cdots ba \cdots, \quad (8)$$

where  $(a, b)$  is any pair of symbols in  $\mathcal{A}$  different from  $(i, i \pm 1)$  or  $(\bar{i}, \bar{i} \pm 1)$ . (This statement is a special case of Tits’s theorem [45], for the Coxeter group  $S_n \times S_n \times (S_2)^n$ .)

In view of Theorem 13, it suffices to show that if Theorem 23 holds for some factorization scheme  $\mathbf{i}$ , then it also holds for any factorization scheme  $\mathbf{i}'$  obtained from  $\mathbf{i}$  by one of the transformations (7) and (8). To see this, it is enough to demonstrate that the collections of parameters  $\{t_k\}$  and  $\{t'_k\}$  in the equality

$$x_{i_1}(t_1) \cdots x_{i_{n^2}}(t_{n^2}) = x_{i'_1}(t'_1) \cdots x_{i'_{n^2}}(t'_{n^2})$$

are related to each other by (invertible) subtraction-free rational transformations. The latter is a direct consequence of the commutation relations between elementary Jacobi matrices, which can be found in [15, Section 2.2 and (4.17)]. The most important of these relations are the following.

First, for  $i = 1, \dots, n - 1$  and  $j = i + 1$ , we have

$$x_i(t_1) x_{\mathcal{D}}(t_2) x_{\mathcal{D}}(t_3) x_{\bar{i}}(t_4) = x_{\bar{i}}(t'_1) x_{\mathcal{D}}(t'_2) x_{\mathcal{D}}(t'_3) x_i(t'_4),$$

where

$$t'_1 = \frac{t_3 t_4}{T}, \quad t'_2 = T, \quad t'_3 = \frac{t_2 t_3}{T}, \quad t'_4 = \frac{t_1 t_3}{T}, \quad T = t_2 + t_1 t_3 t_4.$$

The proof of this relation (which is the only nontrivial relation associated with (8)) amounts to verifying that

$$\begin{bmatrix} 1 & t_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_2 & 0 \\ 0 & t_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t'_1 & 1 \end{bmatrix} \begin{bmatrix} t'_2 & 0 \\ 0 & t'_3 \end{bmatrix} \begin{bmatrix} 1 & t'_4 \\ 0 & 1 \end{bmatrix}.$$

Also, for any  $i$  and  $j$  such that  $|i - j| = 1$ , we have the following relation associated with (7):

$$\begin{aligned} x_i(t_1) x_j(t_2) x_i(t_3) &= x_j(t'_1) x_i(t'_2) x_j(t'_3), \\ x_{\bar{i}}(t_1) x_j(t_2) x_{\bar{i}}(t_3) &= x_j(t'_1) x_{\bar{i}}(t'_2) x_j(t'_3), \end{aligned}$$

where

$$t'_1 = \frac{t_2 t_3}{T}, \quad t'_2 = T, \quad t'_3 = \frac{t_1 t_2}{T}, \quad T = t_1 + t_3.$$

One sees that in the commutation relations above, the formulas expressing the  $t'_k$  in terms of the  $t_l$  are indeed subtraction-free.  $\square$

Theorem 23 suggests an alternative approach to total positivity criteria via the following *factorization problem*: for a given factorization scheme  $\mathbf{i}$ , find the genericity conditions on a matrix  $x$  assuring that  $x$  can be factored as

$$x = x_{\mathbf{i}}(t_1, \dots, t_{n^2}) = x_{i_1}(t_1) \cdots x_{i_{n^2}}(t_{n^2}), \quad (9)$$

and compute explicitly the factorization parameters  $t_k$  as functions of  $x$ . Then, the total positivity of  $x$  will be equivalent to the positivity of all these functions. Note that the criterion in Theorem 9 was essentially obtained in this way: for the factorization scheme  $\mathbf{i}_{\max}$ , the factorization parameters  $t_k$  are Laurent monomials in the initial minors of  $x$  (cf. Lemma 6).

A complete solution of the factorization problem for an arbitrary factorization scheme was given in [15, Theorems 1.9 and 4.9]. An interesting (and unexpected) feature of this solution is that, in general, the  $t_k$  are not Laurent monomials in the minors of  $x$ ; the word  $\mathbf{i}_{\max}$  is quite exceptional in this respect. It turns out, however, that the  $t_k$  are Laurent monomials in the minors of another matrix  $x'$  obtained from  $x$  by the following birational transformation:

$$x' = [x^T w_0]_+ w_0 (x^T)^{-1} w_0 [w_0 x^T]_-. \quad (10)$$

Here,  $x^T$  denotes the transpose of  $x$ , and  $w_0$  is the permutation matrix with 1’s on the antidiagonal; finally,  $y = [y]_- [y]_0 [y]_+$  denotes the Gaussian (LDU) decomposition of a square matrix  $y$  provided such a decomposition exists.

In the special cases  $n = 2$  and  $n = 3$ , the transformation  $x \mapsto x'$  is given by

$$x' = \begin{bmatrix} x_{11} x_{12}^{-1} x_{21}^{-1} & x_{21}^{-1} \\ x_{12}^{-1} & x_{22} \det(x)^{-1} \end{bmatrix}$$

and

$$x' = \begin{bmatrix} \frac{x_{11}}{x_{31} x_{13}} & \frac{\Delta_{12,13}}{x_{31} \Delta_{12,23}} & \frac{1}{x_{31}} \\ \frac{\Delta_{13,12}}{x_{13} \Delta_{23,12}} & \frac{x_{33} \Delta_{12,12} - \det(x)}{\Delta_{23,12} \Delta_{12,23}} & \frac{x_{32}}{\Delta_{23,12}} \\ \frac{1}{x_{13}} & \frac{x_{23}}{\Delta_{12,23}} & \frac{\Delta_{23,23}}{\det(x)} \end{bmatrix}.$$

The following theorem provides an alternative explanation for the family of total positivity criteria in Theorem 16.

**THEOREM 24** [15]. *The right-hand side of (10) is well defined for any  $x \in G_{>0}$ ; moreover, the “twist map”  $x \mapsto x'$  restricts to a bijection of  $G_{>0}$  with itself.*

*Let  $x$  be a totally positive  $n \times n$  matrix, and  $\mathbf{i}$  a factorization scheme. Then, the parameters  $t_1, \dots, t_{n^2}$  appearing in (9) are related by an invertible monomial transformation to the  $n^2$  chamber minors (for the double wiring diagram associated with  $\mathbf{i}$ ) of the twisted matrix  $x'$  given by (10).*

In [15], we explicitly describe the monomial transformation in Theorem 24, as well as its inverse, in terms of the combinatorics of the double wiring diagram.

## Double Bruhat Cells

Our presentation in this section will be a bit sketchy; details can be found in [15].

Theorem 23 provides a family of bijective (and biregular) parametrizations of the totally positive variety  $G_{>0}$  by  $n^2$ -tuples of positive real numbers. The totally non-negative variety  $G_{\geq 0}$  is much more complicated (note that the map in Theorem 14 is surjective but not injective). In this section, we show that  $G_{\geq 0}$  splits naturally into “simple pieces” corresponding to pairs of permutations from  $S_n$ .

**THEOREM 25** [15]. *Let  $x \in G_{\geq 0}$  be a totally non-negative matrix. Suppose that a word  $\mathbf{i}$  in the alphabet  $\mathcal{A}$  is such that  $x$  can be factored as  $x = x_{\mathbf{i}}(t_1, \dots, t_m)$  with positive  $t_1, \dots, t_m$ , and  $\mathbf{i}$  has the smallest number of uncircled entries among all words with this property. Then, the subword of  $\mathbf{i}$  formed by entries from  $\{1, \dots, n-1\}$  (resp. from  $\{1, \dots, n-1\}$ ) is a reduced word for some permutation  $u$  (resp.  $v$ ) in  $S_n$ . Furthermore, the pair  $(u, v)$  is uniquely determined by  $x$  (i.e., does not depend on the choice of  $\mathbf{i}$ ).*

In the situation of Theorem 25, we say that  $x$  is of *type*  $(u, v)$ . Let  $G_{>0}^{u,v} \subset G_{\geq 0}$  denote the subset of all totally non-negative matrices of type  $(u, v)$ ; thus,  $G_{\geq 0}$  is the disjoint union of these subsets.

Every subvariety  $G_{>0}^{u,v}$  has a family of parametrizations similar to those in Theorem 23. Generalizing Definition 22, let us call a word  $\mathbf{i}$  in the alphabet  $\mathcal{A}$  a *factorization scheme of type*  $(u, v)$  if it contains each circled entry  $\textcircled{i}$  exactly once, and the barred (resp. unbarred) entries of  $\mathbf{i}$  form a reduced word for  $u$  (resp.  $v$ ); in particular,  $\mathbf{i}$  is of length  $\ell(u) + \ell(v) + n$ .

**THEOREM 26** [15]. *For an arbitrary factorization scheme  $\mathbf{i}$  of type  $(u, v)$ , the product map  $x_{\mathbf{i}}$  restricts to a bijection between  $(\ell(u) + \ell(v) + n)$ -tuples of positive real numbers and totally non-negative matrices of type  $(u, v)$ .*

Comparing Theorems 26 and 23, we see that

$$G_{>0}^{w_0, w_0} = G_{>0}; \quad (11)$$

that is, the totally positive matrices are exactly the totally non-negative matrices of type  $(w_0, w_0)$ .

We now show that the splitting of  $G_{\geq 0}$  into the union of varieties  $G_{>0}^{u,v}$  is closely related to the well-known *Bruhat decompositions* of the general linear group  $G = GL_n$ . Let  $B$  (resp.  $B_-$ ) denote the subgroup of upper-triangular (resp. lower-triangular) matrices in  $G$ . Recall (see, e.g., [1, §4]) that each of the double coset spaces  $BG/B$  and  $B_- \backslash G/B_-$  has cardinality  $n!$ , and one can choose the permutation matrices  $w \in S_n$  as their common representatives. To every two permutations  $u$  and  $v$  we associate the *double Bruhat cell*  $G^{u,v} = BuB \cap B_-vB_-$ ; thus,  $G$  is the disjoint union of the double Bruhat cells.

Each set  $G^{u,v}$  can be described by equations and inequalities of the form  $\Delta(x) = 0$  and/or  $\Delta(x) \neq 0$ , for some collection of minors  $\Delta$ . (See [15, Proposition 4.1] or [16].) In particular, the open double Bruhat cell  $G^{w_0, w_0}$  is given by

nonvanishing of all “antiprincipal” minors  $\Delta_{[1,i],[n-i+1,n]}(x)$  and  $\Delta_{[n-i+1,n],[1,i]}(x)$  for  $i = 1, \dots, n-1$ .

**THEOREM 27** [15]. *A totally non-negative matrix is of type  $(u, v)$  if and only if it belongs to the double Bruhat cell  $G^{u,v}$ .*

In view of (11), Theorem 27 provides the following simple test for total positivity of a totally non-negative matrix.

**COROLLARY 28** [23]. *A totally non-negative matrix  $x$  is totally positive if and only if  $\Delta_{[1,i],[n-i+1,n]}(x) \neq 0$  and  $\Delta_{[n-i+1,n],[1,i]}(x) \neq 0$  for  $i = 1, \dots, n$ .*

The results obtained above for  $G_{>0}^{w_0, w_0} = G_{>0}$  (as well as their proofs) extend to the variety  $G_{>0}^{u,v}$  for an arbitrary pair of permutations  $u, v \in S_n$ . In particular, the factorization schemes for  $(u, v)$  (or rather their uncircled parts) can be visualized by *double wiring diagrams of type*  $(u, v)$  in the same way as before, except now any two pseudolines intersect at most once, and the lines are permuted “according to  $u$  and  $v$ .” Every such diagram has  $\ell(u) + \ell(v) + n$  chamber minors, and their positivity provides a criterion for a matrix  $x \in G^{u,v}$  to belong to  $G_{>0}^{u,v}$ . The factorization problem and its solution provided by Theorem 24 extend to any double Bruhat cell, with an appropriate modification of the twist map  $x \mapsto x'$ . The details can be found in [15].

If the double Bruhat cell containing a matrix  $x \in G$  is not specified, then testing  $x$  for total non-negativity becomes a much harder problem; in fact, every known criterion involves exponentially many (in  $n$ ) minors. (See [8] for related complexity results.) The following corollary of a result by Cryer [10] was given by Gasca and Peña [24].

**THEOREM 29.** *An invertible square matrix is totally non-negative if and only if all its minors occupying several initial rows or several initial columns are non-negative, and all its leading principal minors are positive.*

This criterion involves  $2^{n+1} - n - 2$  minors, which is roughly the square root of the total number of minors. We do not know whether this criterion is optimal.

## Oscillatory Matrices

We conclude the article by discussing the intermediate class of oscillatory matrices that was introduced and intensively studied by Gantmacher and Krein [20, 22]. A matrix is *oscillatory* if it is totally non-negative while some power of it is totally positive; thus, the set of oscillatory matrices contains  $G_{>0}$  and is contained in  $G_{\geq 0}$ . The following theorem provides several equivalent characterizations of oscillatory matrices; the equivalence of (a)–(c) was proved in [22], and the rest of the conditions were given in [17].

**THEOREM 30** [17,22]. *For an invertible totally non-negative  $n \times n$  matrix  $x$ , the following are equivalent:*

- (a)  $x$  is oscillatory;
- (b)  $x_{i,i+1} > 0$  and  $x_{i+1,i} > 0$  for  $i = 1, \dots, n-1$ ;

- (c)  $x^{n-1}$  is totally positive;  
 (d)  $x$  is not block-triangular (cf. Figure 11);

$$\begin{bmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \quad \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{bmatrix}$$

Figure 11. Block-triangular matrices.

- (e)  $x$  can be factored as  $x = x_i(t_1, \dots, t_i)$ , for positive  $t_1, \dots, t_i$  and a word  $\mathbf{i}$  that contains every symbol of the form  $i$  or  $\bar{i}$  at least once;  
 (f)  $x$  lies in a double Bruhat cell  $G^{u,v}$ , where both  $u$  and  $v$  do not fix any set  $\{1, \dots, i\}$ , for  $i = 1, \dots, n - 1$ .

*Proof.* Obviously, (c)  $\Rightarrow$  (a)  $\Rightarrow$  (d). Let us prove the equivalence of (b), (d), and (e). By Theorem 12,  $x$  can be represented as the weight matrix of some planar network  $\Gamma(\mathbf{i})$  with positive edge weights. Then, (b) means that sink  $i + 1$  (resp.  $i$ ) can be reached from source  $i$  (resp.  $i + 1$ ), for all  $i$ ; (d) means that for any  $i$ , at least one sink  $j > i$  is reachable from a source  $h \leq i$ , and at least one sink  $h \leq i$  is reachable from a source  $j > i$ ; and (e) means that  $\Gamma(\mathbf{i})$  contains positively and negatively sloped edges connecting

any two consecutive levels  $i$  and  $i + 1$ . These three statements are easily seen to be equivalent.

By Theorem 27, (e)  $\Leftrightarrow$  (f). It remains to show that (e)  $\Rightarrow$  (c). In view of Theorem 26 and (11), this can be restated as follows: given any permutation  $\mathbf{j}$  of the entries  $1, \dots, n - 1$ , prove that the concatenation  $\mathbf{j}^{n-1}$  of  $n - 1$  copies of  $\mathbf{j}$  contains a reduced word for  $w_0$ . Let  $\mathbf{j}'$  denote the subsequence of  $\mathbf{j}^{n-1}$  constructed as follows. First,  $\mathbf{j}'$  contains all  $n - 1$  entries of  $\mathbf{j}^{n-1}$  which are equal to  $n - 1$ . Second,  $\mathbf{j}'$  contains the  $n - 2$  entries equal to  $n - 2$  which interlace the  $n - 1$  entries chosen at the previous step. We then include  $n - 3$  interlacing entries equal to  $n - 3$ , and so forth. The resulting word  $\mathbf{j}'$  of length  $\binom{n}{2}$  will be a reduced word for  $w_0$ , for it will be equivalent, under the transformations (8), to the lexicographically maximal reduced word  $(n - 1, n - 2, n - 1, n - 3, n - 2, n - 1, \dots)$ .  $\square$

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#### REFERENCES

1. J.L. Alperin and R.B. Bell, *Groups and Representations*, Springer-Verlag, New York, 1995.

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2. T. Ando, "Totally positive matrices," *Linear Alg. Appl.* 90 (1987), 165–219.
3. A. Berenstein, S. Fomin, and A. Zelevinsky, "Parametrizations of canonical bases and totally positive matrices," *Adv. Math.* 122 (1996), 49–149.
4. A. Berenstein and A. Zelevinsky, "Total positivity in Schubert varieties," *Comment. Math. Helv.* 72 (1997), 128–166.
5. F. Brenti, "The applications of total positivity to combinatorics, and conversely," in [25], pp. 451–473.
6. F. Brenti, "Combinatorics and total positivity," *J. Combin. Theory, Ser. A* 71 (1995), 175–218.
7. R.A. Brualdi and H.J. Ryser, "Combinatorial matrix theory," *Encyclopedia of Mathematics and its Applications, Vol. 39*, Cambridge University Press, Cambridge, 1991.
8. A. Cohen, N. Linial, and Yu. Rabinovich, "Totally nonnegative matrices and planar arithmetic circuits," preprint.
9. C. Cryer, "The  $LU$ -factorization of totally positive matrices," *Linear Alg. Appl.* 7 (1973), 83–92.
10. C. Cryer, "Some properties of totally positive matrices," *Linear Alg. Appl.* 15 (1976), 1–25.
11. E. Curtis, D.V. Ingerman, and J. Morrow, "Circular planar graphs and resistor networks," *Linear Alg. Appl.* 283 (1998), 115–150.
12. C.L. Dodgson, *The Mathematical Pamphlets of Charles Lutwidge Dodgson and Related Pieces*, ed. by F.F. Abeles, Lewis Carroll Society of North America, Silver Spring, MD, 1994.
13. A. Edrei, "On the generating function of totally positive sequences, II," *J. Anal. Math.* 2 (1952), 104–109.
14. M. Fekete, "Über ein Problem von Laguerre," *Rend. Circ. Mat. Palermo* 34 (1912), 89–100, 110–120.
15. S. Fomin and A. Zelevinsky, "Double Bruhat cells and total positivity," *J. Am. Math. Soc.* 12 (1999), 335–380.
16. S. Fomin and A. Zelevinsky, "Recognizing Schubert cells," preprint math. CO/9807079, July 1998.
17. S. Fomin and A. Zelevinsky, "Totally nonnegative and oscillatory elements in semisimple groups," preprint math. RT/9811100, November 1998.
18. W. Fulton and J. Harris, *Representation Theory*, Springer-Verlag, New York, 1991.
19. D. Gale, "Mathematical entertainments: The strange and surprising saga of the Somos sequences," *Math. Intelligencer* 13 (1991), no. 1, 40–42.
20. F.R. Gantmacher and M.G. Krein, "Sur les matrices oscillatoires," *C. R. Acad. Sci. (Paris)* 201 (1935), 577–579.
21. F.R. Gantmacher and M.G. Krein, "Sur les matrices complètement non négatives et oscillatoires," *Compos. Math.* 4 (1937), 445–476.
22. F.R. Gantmacher and M.G. Krein, *Oszillationsmatrizen, Oszillationskerne und Kleine Schwingungen Mechanischer Systeme*, Akademie-Verlag, Berlin, 1960. (Russian edition: Moscow-Leningrad, 1950.)
23. M. Gasca and J.M. Peña, "Total positivity and Neville elimination," *Linear Alg. Appl.* 165 (1992), 25–44.
24. M. Gasca and J.M. Peña, "Total positivity, QR factorization, and Neville elimination," *SIAM J. Matrix Anal. Appl.* 14 (1993), 1132–1140.
25. M. Gasca and C.A. Micchelli (eds.), *Total Positivity and Its Applications, Mathematics and Its Applications* No. 359, Kluwer Academic Publishers, Dordrecht, 1996.
26. I. Gessel and G.X. Viennot, "Binomial determinants, paths, and hooklength formulae," *Adv. Math.* 58 (1985), 300–321.
27. I. Gessel and G.X. Viennot, "Determinants, paths, and plane partitions," preprint.
28. S. Karlin, *Total Positivity*, Stanford University Press, Stanford, CA, 1968.
29. S. Karlin and G. McGregor, "Coincidence probabilities," *Pacific J. Math.* 9 (1959), 1141–1164.
30. B. Lindström, "On the vector representations of induced matroids," *Bull. London Math. Soc.* 5 (1973), 85–90.
31. P. Littelmann, "Bases canoniques et applications," *Séminaire Bourbaki* Vol. 1997/98, Astérisque No. 252 (1998), Exp. No. 847, 5, 287–306.
32. C. Loewner, "On totally positive matrices," *Math. Z.* 63 (1955), 338–340.
33. G. Lusztig, "Total positivity in reductive groups," *Lie Theory and Geometry: in Honor of Bertram Kostant*, Progress in Mathematics 123, Birkhäuser, Boston 1994, pp. 531–568.
34. G. Lusztig, "Introduction to total positivity," *Positivity in Lie theory: Open Problems*, de Gruyter Expositions in Mathematics 26, de Gruyter, Berlin, 1998, pp. 133–145.
35. J.L. Malouf, "An integer sequence from a rational recursion," *Discrete Math.* 110 (1992), 257–261.
36. E.H. Moore, "Concerning the abstract groups of order  $k!$  and  $\frac{1}{2}k!$ ," *Proc. London Math. Soc.* 28 (1897), 357–366.
37. T. Muir, *The Theory of Determinants*, 2nd ed., Macmillan, London, 1906, Vol. 1.
38. A. Pinkus, "Spectral properties of totally positive kernels and matrices," in [25], pp. 477–511.
39. G. Ringel, "Teilungen der Ebene durch Geraden oder topologische Geraden," *Math. Z.* 64 (1956), 79–102.
40. I.J. Schoenberg, *Selected Papers*, Birkhäuser, Boston 1988, Vol. 2.
41. I.J. Schoenberg, "Über Variationsvermindernde lineare Transformationen," *Math. Z.* 32 (1930), 321–328.
42. R.P. Stanley, "Log-concave and unimodal sequences in algebra, combinatorics and geometry," *Ann. NY Acad. Sci.* 576 (1989), 500–534.
43. J.R. Stembridge, "Immanants of totally positive matrices are non-negative," *Bull. London Math. Soc.* 23 (1991), 422–428.
44. E. Thoma, "Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen symmetrischen Gruppe," *Math. Z.*, 85 (1964), 40–61.
45. J. Tits, "Le problème des mots dans les groupes de Coxeter," 1969 *Symposia Mathematica* (INDAM, Rome, 1967/68), Vol. 1., 175–185, Academic Press, London.
46. A.M. Whitney, "A reduction theorem for totally positive matrices," *J. Anal. Math.* 2 (1952), 88–92.