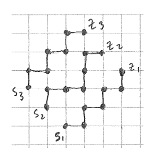
ENUMERATIVE COMBINATORICS (SF2708) 2012 PROBLEM SET 2

*To be handed in no later than* **March 20**. *You may cooperate but you must write your solutions by yourselves. Please write full proofs!*

(1) Exercise 1.122 of Stanley,
(2) Exercise 1.184 a of Stanley,
(3) Let $m, n \geq 1$ be integers. Prove the identity
$$\sum_{k=0}^{n} \binom{n}{k} k^{n-k} |\{\pi \in \mathfrak{S}_k \mid \pi^m = id\}| = |\{f : [n] \to [n] \mid f^{m+1} = f\}|.$$
Here $f^i = f \circ f \circ \cdots \circ f$, $i$ times. I suggest you try the case $m = 1$ first and then generalize.
(4) Define the *drop set* of a permutation $\pi \in \mathfrak{S}_n$ to be $\mathrm{Dr}(\pi) = \{i : \pi(i) < i\}$. Let $\gamma_n : 2^{[n]} \to \mathbb{N}$ be defined by $\gamma_n(S) = |\{\pi \in \mathfrak{S}_n : \mathrm{Dr}(\pi) = S\}|$. For $S = \{s_1 < \cdots < s_k\} \subseteq [n]$, let $\eta_n(S) = (n-k)!(s_1 - 1) \cdots (s_k - k)$. Prove that
$$\gamma_n(S) = \sum_{T \supseteq S} (-1)^{|T \setminus S|} \eta_n(T).$$
(5) Let $k$ and $n$ be positive integers. Prove that all minors of the matrix (with binomial numbers)
$$\left( \binom{k+i}{k+j-i} \right)_{0 \leq i,j \leq n}$$
are nonnegative. Here $\binom{k+i}{k+j-i} = 0$ if $k + j - i < 0$.
(6) Let $B(n, k, \ell)$ be the number of $k \times \ell$ matrices with entries from $\{0, 1, \ldots, n\}$ and such that each column and each row is weakly increasing. Express $B(n, k, \ell)$ as a determinant. *Hint:*
$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$



(7) Exercise 2.25 a,b of Stanley.