To be handed in no later than May 22. You may cooperate but you must write your solutions by yourselves. Please write full proofs!

(1) Exercise 3.62 a–e,
(2) Exercise 3.63 a–c,
(3) Let $P$ be a labeled poset and for $1 \leq k \leq p$, let $e_k(P)$ be the number of surjective $P$-partitions $\sigma : P \to [k]$. Recall that
\[ \Omega_P(x) = \sum_{k=1}^p e_k(P) \binom{x}{k}. \]
Let
\[ E_P(x) = \sum_{k=1}^p e_k(P)x^k. \]
(a) Prove that $E_P(x) = (1 + x)^p A_P(x/(1 + x)).$
(b) Suppose that $P$ is naturally labeled and that $P$ is not an anti-chain. Let $M$ be the set of maximal elements of $P$. Prove that
\[ E_P(x) = (1 + x) \sum_{\emptyset \subset S \subseteq M} (-1)^{|S|+1} E_{P \setminus S}(x). \]
\[ \text{Hint. Prove first that for } 2 \leq k \leq p, \]
\[ e_k(P) = \sum_{\emptyset \subset I \subset P} e_{k-1}(I), \]
where the sum is over all non-empty proper order ideals $I$ of $P$.
(c) Deduce a recursion similar to (1) for $A_P(x)$.
(4) Let $P$ be a finite poset. Let $a(P)$ denote the minimum number of anti-chains required to partition $P$ into anti-chains. Prove that $a(P) = \ell(P) + 1$, where $\ell(P)$ is the length of $P$, see page 244.
(5) Let $G = (V, E)$ be a finite graph, and recall that the chromatic polynomial of $G$ is defined by
\[ \chi_G(n) = |\{f : V \to [n] \mid f(i) \neq f(j) \text{ whenever there is an edge between } i \text{ and } j\}|. \]
Define a polynomial $W_G(x) = \sum_{k=0}^{|V|} a_k(G)x^k$ by
\[ \sum_{n=0}^{\infty} \chi_G(n)x^n = \frac{W_G(x)}{(1 - x)^{|V|+1}}. \]
Prove that $a_k(G) \geq 0$ for all $0 \leq k \leq |V|$. \textit{Hint.} Express $\chi_G(n)$ as a sum of order polynomials.