To be handed in no later than **May 22**. *You may cooperate but you must write your solutions by yourselves. Please write full proofs!*

(1) Exercise 3.62 a–e,
(2) Exercise 3.63 a–c,
(3) Let P be a labeled poset and for $1 \leq k \leq p$, let $e_k(P)$ be the number of surjective P-partitions $\sigma : P \to [k]$. Recall that

$$\Omega_P(x) = \sum_{k=1}^{p} e_k(P) \binom{x}{k}.$$

Let

$$E_P(x) = \sum_{k=1}^{p} e_k(P) x^k.$$

(a) Prove that $E_P(x) = (1+x)^p A_P(x/(1+x))$.
(b) Suppose that P is naturally labeled and that P is not an anti-chain. Let M be the set of maximal elements of P. Prove that

$$E_P(x) = (1+x) \sum_{\emptyset \subset S \subseteq M} (-1)^{|S|+1} E_{P \setminus S}(x). \tag{1}$$

Hint. Prove first that for $2 \leq k \leq p$,

$$e_k(P) = \sum_{\emptyset \subset I \subset P} e_{k-1}(I),$$

where the sum is over all non-empty proper order ideals I of P.
(c) Deduce a recursion similar to (1) for $A_P(x)$.

(4) Let P be a finite poset. Let $a(P)$ denote the minimum number of anti-chains required to partition P into anti-chains. Prove that $a(P) = \ell(P) + 1$, where $\ell(P)$ is the length of P, see page 244.
(5) Let $G = (V, E)$ be a finite graph, and recall that the chromatic polynomial of G is defined by

$$\chi_G(n) = |\{f : V \to [n] \mid f(i) \neq f(j) \text{ whenever there is an edge between } i \text{ and } j\}|$$

Define a polynomial $W_G(x) = \sum_{k=0}^{|V|} a_k(G) x^k$ by

$$\sum_{n=0}^{\infty} \chi_G(n) x^n = \frac{W_G(x)}{(1-x)^{|V|+1}}.$$

Prove that $a_k(G) \geq 0$ for all $0 \leq k \leq |V|$. *Hint.* Express $\chi_G(n)$ as a sum of order polynomials.