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Note

Permutations with one or two 132-subsequences

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Abstract

We prove a strikingly simple formula for the number of permutations containing exactly one subsequence of type 132. We show that this number equals the number of partitions of a convex $(n+1)$-gon into $n-2$ parts by noncrossing diagonals. We also prove a recursive formula for the number d_n of those containing exactly two such subsequences, yielding that $\{d_n\}$ is P-recursive.

1. Introduction

Let $q=(q_1,q_2,\ldots,q_k)\in S_k$ be a permutation, and let $k\leqslant n$. We say that the permutation $p=(p_1,p_2,\ldots,p_n)\in S_n$ contains a subsequence (or pattern) of type q if and only if there is a set of indices$1\leqslant i_{q_1}<i_{q_2}<\cdots<i_{q_k}\leqslant n$ such that $p(i_1)<p(i_2)<\cdots<p(i_k)$. Otherwise we say that p is q-avoiding.

Thus a permutation is 12345-avoiding if it does not contain any increasing subsequence of length 5 in the above one-line notation. For another example, a permutation is 132-avoiding if it doesn't contain three elements among which the leftmost is the smallest and the middle one is the largest.

In 1990 Herb Wilf asked the following question: how many permutations of length n (or n-permutations) do avoid a given subsequence q? Much effort has been made to compute or at least estimate this number $S_n(q)$.

The case when the length of q is 3 was solved in [7] by proving that $S_n(q)=c_n$, the nth Catalan number for any 3-permutation q. Bijective results were proven in [10]. The case of length 4 was handled in [1]. However, the general problem is still open. To illustrate the complexity of the problem, we note that the problem of counting permutations in S_n avoiding q is known to be #P-complete. Thus in the general case all we can hope for is an upper bound or an asymptotical formula for this number, not an exact formula. Similarly, the problem of deciding whether a given permutation contains a given subsequence is NP-complete. See [2] for these results.

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Recently, attention has been paid to the problem of counting the number of permutations of length n containing a *given number* r (as opposed to 0) of subsequences of a certain type q. Noonan [5] has proved that the number of permutations in S_n containing exactly one subsequence of type 123 is $\frac{3}{n}\binom{2n}{n-3}$, a surprisingly simple formula. It is an interesting problem to describe how these formulae change if q is kept fixed and r increases.

In this paper we solve the same problem for the nonmonotonic permutations of length 3. By a generating function argument we show that the number of permutations of length n containing exactly one subsequence of type 132 is $\binom{2n-3}{n-3}$. This formula is even simpler than the one cited above and asks for a direct combinatorial proof. It is even more surprising that this formula is simpler than that for the number of 132-avoiding permutations, namely $S_n(132)=\binom{2n}{n}/(n+1)$. Our result implies the same formula for the number of permutations containing exactly one subsequence of type 231, 213, or 312, respectively, and thus it completely arranges the problem for all subsequences of length 3. This formula was conjectured by Zeilberger and Noonan in [6].

We also provide an other combinatorial interpretation of these numbers $b_n=\binom{2n-3}{n-3}$ by showing that they are equal to the number of partitions of a convex $(n+1)$-gon into $n-2$ parts by noncrossing diagonals. This is very similar to the well-known interpretation of the Catalan numbers as the number of partitions of a convex $(n+2)$-gon into n parts by noncrossing diagonals.

Then we turn to permutations containing exactly two 132-patterns. Let d_n be the number of such n-permutations. We are going to show that the sequence $\{d_n\}$ is P-recursive, that is, d_n-can be obtained as a linear combination of $d_{n-1},\dots,d_{n-k}$ (for some fixed k) with polynomial coefficients. This is equivalent to saying that its ordinary generating function $D(x)$ is d-finite, that is, the dimension of the complex vector space spanned by $D'(x), D''(x),\dots$ is finite.

The author believes that a similar result is true in general, that is, the number of n-permutations containing exactly r copies of the pattern 132 is a P-recursive function of n. However, one will clearly need new methods to prove this since checking all cases becomes harder and harder as r grows. This paper illustrates the increase in complexity while changing r from 1 to 2.

2. Background

This section recalls some basic results concerning Catalan numbers and P-recursive sequences we will need in our proof. We only indicate how they can be proved, details can be found in [7,9,8].

Lemma 1. *The number of permutations avoiding* 132 *is the nth Catalan number,* $c_n=\binom{2n}{n}/(n+1)$. *This also equals the number of partitions of a convex* $(n+2)$*-gon into n triangles by noncrossing diagonals.*

Proof. There are several ways to see this. For example, this is a direct consequence of Corollary 3 below. □

Lemma 2. *The Catalan numbers satisfy the following recursion*: $c_n = \sum_{i=0}^{n-1} c_{i-1}c_{n-i}$ (*with* $c_0 = 1$).

Proof. Observe that $c_{i-1}c_{n-i}$ is the number of n-permutations in which n is in the ith position, and the lemma follows. □

Corollary 3. *Let* $C(x) = \sum_{n=0}^{\infty} c_n x^n$, *the ordinary generating function of the Catalan numbers. Then*

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x}. \tag{1}$$

Proof. The recursive formula of the previous lemma implies $C^2(x)x + 1 = C(x)$ which proves the statement. □

Lemma 4. *A sequence* $\{t_n\}$ *is P-recursive if and only if its ordinary generating function is d-finite. The d-finite power series form a subalgebra of* $\mathbb{C}[[x]]$.

3. The formula for b_n

In the sequel elements of an n-permutation on the left of the entry n will be called *front elements*, whereas those on the right of n will be called *back elements*.

Theorem 5. *Let* b_n *be the number of n-permutations having exactly one subsequence of type* 132. *Then* $b_0 = b_1 = b_2 = 0$ *and for all* $n \geqslant 3$ *we have*

$$b_n = \binom{2n-3}{n-3}. \tag{2}$$

Proof. Take any n-permutation p and suppose that the entry n is in the ith position in p. Then there are three ways p can contain exactly one subsequence S of type 132.

(1) When all elements of S are front entries. Then any front entry must be larger than any back entry for any pair violating this condition would form an additional 132-subsequence with n. Therefore, the i largest entries must be front entries n (these are the entries $n-1, n-2, \ldots, n-i+1$), while the $n-i$ smallest entries must be back entries (these are the entries $1, 2, \ldots, n-i$). Moreover, there can be no subsequence of type 132 formed by back entries. Thus all we can do is to take a 132-avoiding permutation on the $n-i$ back entries in c_{n-i} ways and take a permutation having exactly one 132-subsequence on the $i-1$ front entries. This yields $b_{i-1}c_{n-i}$ permutations of the desired property.

(2) When all elements of S are back entries. The argument of the previous case holds here, too, we must only swap the roles of the front and back entries. Then we get that in this case we have $c_{i-1}b_{n-i}$ permutations of the desired property.

(3) Finally, it can happen that the leftmost element x of S is a front entry and the rightmost element z of S is a back entry. This case is slightly more complicated. Note that here we must have $2 \leqslant i \leqslant n-1$, otherwise either the set of front entries or that of back entries would be empty.

Observe that there is exactly one pair (x,z) so that x is a front entry, z is a back entry and $x<z$. (For any such pair and n form a 132-subsequence.) This implies that the front entries are $n-1, n-2, \ldots, n-i+2, n-i$ and the back entries are $1, 2, \ldots, n-i-1, n-i+1$, the only pair with the given property is $(n-i, n-i+1) = (x,z)$, and any other front entry is larger than both x and z.

Let us take these entries x and z. Clearly, all 132-subsequences of the given type must start with x and must end with z. We claim that the middle entry of S must be n. Indeed, if the middle element were some other w, then xnz and xwz would both be 132-subsequences. (Recall that $x<z$ and they both are smaller than any other front entry.) Moreover, we claim that x must be the rightmost front entry, in other words, it must be in the position directly on the left of n. Indeed, if there were any entry y between x and n, then xyz and xnz would both be 132-subsequences for y is a front entry and thus larger than x and z.

Therefore, all we can do is put the entry $n-i$ in the $(i-1)$st position, then take any 132-avoiding permutation on the first $i-2$ elements in c_{i-2} ways and take any 132-avoiding permutation on the $n-i$ back entries in c_{n-i} ways. This gives us $c_{i-2}c_{n-i}$ permutations of the desired property.

Summing for all permitted i in each of these three cases we get that

$$b_n = \sum_{i=1}^{n-1} b_{i-1}c_{n-i} + \sum_{i=1}^{n-1} c_{i-1}b_{n-i} + \sum_{i=2}^{n-1} c_{i-2}c_{n-i}. \tag{3}$$

Note that the first two sums are equal for they contain the same summands. Moreover, by Lemma 2 we can easily see that the last sum equals $c_{n-1} - c_{n-2}$. Thus the above recursive formula for b_n simplifies to

$$b_n = 2 \cdot \left(\sum_{i=1}^{n-1} b_{i-1}c_{n-i} \right) + c_{n-1} - c_{n-2}. \tag{4}$$

Now let $B(x) = \sum_{n=0}^{\infty} b_n x^n$, the ordinary generating function of the sequence $\{b_n\}$. Then one sees by (4) and by equating the coefficients of x^n that the following functional equation must hold:

$$B(x) = 2xB(x)C(x) + (x - x^2)C(x) - x.$$

This yields

$$B(x) = \frac{C(x)(x-x^2) - x}{1 - 2xC(x)}. \tag{5}$$

Recall that Corollary 3 provides the explicit form of $C(x)$. Plugging it in (5) we get

$$B(x)=\frac{\frac{1-\sqrt{1-4x}}{2}\cdot(1-x)-x}{\sqrt{1-4x}},$$

or, splitting the numerator into three parts,

$$B(x)=\frac{1}{2\cdot(\sqrt{1-4x})}\cdot(1-x)-\frac{x}{\sqrt{1-4x}}+\frac{1-x}{2}. \tag{6}$$

A routine computation (see [9]) yields that $1/\sqrt{1-4x}=\sum_{n\geqslant 0}\binom{2n}{n}x^n$. Therefore (6) is equivalent to

$$\begin{aligned}B(x)&=\frac{1}{2}(1-x)\sum_{n\geqslant 0}\binom{2n}{n}x^n-x\sum_{n\geqslant 0}\binom{2n}{n}x^n+\frac{1-x}{2}\\&=\frac{1}{2}\sum_{n\geqslant 0}\binom{2n}{n}x^n-\frac{1}{2}\sum_{n\geqslant 1}\binom{2n-2}{n-1}x^n-\sum_{n\geqslant 1}\binom{2n-2}{n-1}x^n+\frac{1-x}{2}.\end{aligned}$$

Equating coefficients of x^n, with $2\leqslant n$ we get that

$$\begin{aligned}b_n&=\frac{1}{2}\cdot\binom{2n}{n}-\binom{2n-2}{n-1}-\frac{1}{2}\cdot\binom{2n-2}{n-1}\\&=\binom{2n-1}{n-1}-\binom{2n-2}{n-1}-\binom{2n-3}{n-2}\\&=\binom{2n-2}{n-2}-\binom{2n-3}{n-2}=\binom{2n-3}{n-3},\end{aligned}$$

and the theorem is proved. □

Now it is easy to prove by induction that b_n equals the number of partitions of a convex $(n+1)$-gon into $n-2$ parts by noncrossing diagonals. (These partitions were first enumerated by Cayley in [3].) In other words, this is the number of partitions of that polygon into one quadrilateral Q and $n-3$ triangles. Indeed, if $n=3$, the statement is true. Suppose we know the statement for all integers larger than 2 and smaller than n. Let $A_1, A_2, \ldots, A_{n+1}$ denote the vertices of the polygon. Let k be the smallest number so that there is a diagonal A_1A_k in our chosen partition. (If there is no such diagonal, then let $k=n+1$. Thus $3\leqslant k\leqslant n+1$.) The diagonal A_1A_k cuts our polygon into two parts; the part containing the vertex A_2 is called the *front* whereas the other part is called the *back*. Now if the back contains Q (in b_{n-k+2} ways, by the induction hypothesis), then the front is partitioned into triangles in c_{k-3} ways as the diagonal A_2A_k must be contained in our triangulation. If the front contains Q, but A_1 is not a vertex of the quadrilateral, then again, A_2A_k is contained in the partition and we have c_{n-k+1} ways to triangulate the back and then b_{k-2} ways to partition the front. Finally, if A_1 is a vertex of Q, then $Q=A_1A_2A_{k-1}A_k$ and we have c_{k-3} ways to triangulate the rest of the front, in addition to the c_{n-k+1} ways to triangulate the back.

Replacing $k-3$ by i and adding for all i we get the recursive formula of (3) and our claim is proven by induction.

4. Two 132-subsequences

In this section we prove the following theorem.

Theorem 6. *Let d_n be the number of n-permutations containing exactly two subsequences of type* 132. *Then the sequence $\{d_n\}$ is P-recursive.*

Proof. We are going to distinguish three cases, according to the number of *bad pairs*, that is, pairs (x, y) where x is a front entry, y is a back entry and $x<y$. Clearly, each bad pair forms a 132-pattern with n, thus there can be at most two bad pairs. In each case, we are going to go through all possible permutations according to the position of n, but for shortness, we are not announcing it each time.

(1) If there is no bad pair, then by an argument similar to that of the previous section, we see that there are

$$\sum_{i=0}^{n-1} d_{i-1}c_{n-i} + \sum_{i=0}^{n-1} c_{i-1}d_{n-i} + \sum_{i=0}^{n-1} b_{i-1}b_{n-i} = 2\cdot\sum_{i=0}^{n-1} c_{i-1}d_{n-i} + \sum_{i=0}^{n-1} b_{i-1}b_{n-i} \tag{7}$$

permutations containing exactly two 132-patterns.

(2) If there is one bad pair (x, y), then there are two subcases. Note that x, n and y form a 132-pattern, thus we must have exactly one more pattern of that kind. Furthermore, note that the position of n determines our only choice for the pair (x, y).

- If x is directly on the left of n, then we can proceed as in the previous section and see that there are

$$\sum_{i=2}^{n-1} c_{i-2}b_{n-i} + \sum_{i=2}^{n-1} b_{i-2}c_{n-i} = 2\cdot\sum_{i=2}^{n-1} c_{i-2}b_{n-i} = b_{n-1} - c_{n-2} + c_{n-3} \tag{8}$$

permutations with the required property. (The last equality follows from (4).)

- If there are some entries between x and n, then note that each such entry would form a 132-pattern with x and y, thus there can be only one such entry, and therefore x must be exactly two positions to the left of n. There are no other restrictions and there can be no more 132-patterns, which yields

$$\sum_{i=3}^{n-1} c_{i-2}c_{n-i} = c_{n-1} - 2c_{n-2} \tag{9}$$

permutations with the required property. (n must be preceded by at least two elements.)

(3) If there are two bad pairs, then there are two subcases again. Note that the two bad pairs with n provide the two 132-subsequences, thus there cannot be any additional subsequence of that type.

- If the two bad pairs are (x,y) and (x,z), then x must be directly on the left of n to avoid additional 132-patterns. (Again, x, y and z are determined by the position of n.) To ensure that no additional 132-patterns are formed, y and z must be in increasing order, otherwise x completes them to a 132-pattern. Observe that y and z are the largest two back elements, so we simply need to compute the number of 132-avoiding permutations on $n-i$ elements in which the largest two elements are in increasing order. It is easy to see that this number is $c_{n-i}-c_{n-i-1}$, since the only way those two elements can be in decreasing order is when the larger one is in the leftmost position of the permutation. Therefore, in this subcase we have

$$\sum_{i=2}^{n-2} c_{i-2}(c_{n-i} - c_{n-i-1}) = c_{n-1} - 2c_{n-2} \tag{10}$$

suitable permutations. (Note that n must precede at least two elements.)

- Finally, if the two bad pairs are (x,z) and (y,z), then x, y and z are again determined by the position of n. In order to avoid additional 132-patterns, x and y must be in the two positions directly on the left of n. They can be in either order as they are smaller than everything on their left and larger than everything on their right, except n and z. There are no other restrictions and no other 132-patterns, thus this subcase gives us

$$2\sum_{i=3}^{n-1} c_{i-2}c_{n-i} = 2c_{n-1} - 4c_{n-2} \tag{11}$$

suitable permutations.

Summing for all cases we get that

$$d_n = 2\cdot\sum_{i=1}^{n-1} c_{i-1}d_{n-i} + \sum_{i=1}^{n-1} b_{i-1}b_{n-i} + b_{n-1} + 4c_{n-1} - 9c_{n-2} + c_{n-3}. \tag{12}$$

It is well known that the sum and the convolution of two P-recursive sequences is P-recursive. Thus, if h_n denotes the sum of all the terms on the right-hand side of (12) except the first one, then h_n is P-recursive. Let $D(x)$ and $H(x)$ be the ordinary generating functions for $\{d_n\}$ and $\{h_n\}$, respectively. Then (12) gives rise to the functional equation

$$D(x) = 2\cdot(C(x)D(x)x + 4x^4) + H(x),$$

that is,

$$D(x) = \frac{H(x) + 4x^4}{1 - 2xC(x)}, \tag{13}$$

thus $D(x)$ is d-finite as $1/(1 - 2xC(x)) = \sqrt{1 - 4x}$ is d-finite. (In fact, it is easy to see that $D(x)$ is even algebraic.) Therefore, d_n is P-recursive, which was to be proved. □

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