KTH Teknikvetenskap

SF2729 Groups and Rings
Suggested solutions to midterm exam
Monday, March 15, 2010

---

(1) a) Show directly from the group axioms that the unit of a group is unique and that there are the cancellation rules

$$ab = ac \Rightarrow b = c \quad \text{and} \quad ac = bc \Rightarrow a = b.$$

(2)

b) Show that the symmetry group of the regular tetrahedron is isomorphic to A_4. **(2)**

c) Show that the alternating group A_n for $n \geq 3$ is generated by the 3-cycles

$$(1\,2\,3), (2\,3\,4), \ldots, (n-2\,n-1\,n).$$

(2)

SOLUTION

a). Let e and e' be units in a group G. Then we have that $ee' = e'$ since e is a (left) unit, but also $ee' = e$ since e' is a (right) unit. Hence $e = e'$ and the unit is unique.

Since there is an inverse to a, we have that

$$ab = ac \Rightarrow a^{-1}(ab) = a^{-1}(ac) \Rightarrow (a^{-1}a)b = (a^{-1}a)b \Rightarrow eb = ec \Rightarrow b = c$$

where we used associativity, that $a^{-1}a = e$ and that e is a unit.

We do similarily with the inverse to c and get

$$ac = bc \Rightarrow (ac)c^{-1} = (bc)c^{-1} \Rightarrow a(cc^{-1}) = b(cc^{-1}) \Rightarrow ae = be \Rightarrow a = b$$

b). Let G be the symmetry group of the tetrahedron. Number the faces $1, 2, 3, 4$. Each symmetry will give a permutation in S_4 in this way. The composition of symmetries corresponds to compositions of the permutations. Hence we have a homomorphism $\Phi : G \longrightarrow S_4$. Since only the identity symmetry preserves all the faces, the kernel is trivial and we deduce that the homomorphism is injective.

No transpositions can be in the image, since there is no rotation that fixes two faces but interchanges the remaining two. On the other hand, all 3-cycles occur in the image, since we have the rotations that fixes one of the vertices of the tetrahedron. Thus the image, which is a subgroup, has to be the alternating group A_4. Since an injective homomorphism is an isomorphism onto its image, we have established that G is isomorphic to A_4.

c). We know that S_n is generated by the adjacent transpositions, $s_1 = (1\,2), s_2 = (2\,3), \ldots, s_{n-1} = (n-1\,n)$. (We can see this by sorting lists.)

A_n are all the even permutations in S_n, which means all the permutations that can be written as a product of an even number of adjacent transpositions. Hence A_n is generated by all the products $s_i s_j$, where $i \neq j$.

We can write the given 3-cycles as products of transpositions in the following way:

$$(i\,i+1\,i+2) = (1\,i+1)(i+1\,i+2) = s_i s_{i+1}$$

Hence we can write

$$\begin{aligned} s_i s_j &= (s_i s_{i+1})(s_{i+1} s_{i+2}) \cdots (s_{j-1} s_j) \\ &= (i\,i+1\,i+2)(i+1\,i+2\,i+3) \cdots (j-1\,j\,j+2) \end{aligned}$$

if $i < j$ and for $i < j$ we have $s_i s_j = (s_j s_i)^{-1}$.

Thus any even permutation is a product of the given 3-cycles.

---

(2) a) Define what it means for a group G to act on a set X and show that such an action gives a group homomorphism $\Phi : G \longrightarrow S_X$, where S_X is the group of bijective functions from X to X under composition. **(2)**

b) Recall that the dihedral group, D_{2n}, can be presented as a factor group of the free group $F[\{r, s\}]$ with the relations $r^n = s^2 = rsrs = 1$. Let X be the set of quadratic complex polynomials $q(x)$ in one variable and let $\xi = e^{2\pi i/n}$. Show that

$$r.q(x) = \xi^{-2} q(\xi^2 x) \quad \text{and} \quad s.q(x) = x^2 q(1/x)$$

defines a well-defined action of D_{2n} on X. **(4)**

SOLUTION

a). An action of a group G on a set X is a function

$$\begin{array}{rcl} G \times X & \longrightarrow & X \\ (a, x) & \longmapsto & g.x \end{array}$$

satisfying

$i)$ $e.x = x$, for all $x \in X$.

$ii)$ $(ab).x = a.(b.x)$, for all $a, b \in G$ and all $x \in X$.

When we have such an action, we get a homomorphism $\Phi : G \longrightarrow S_x$ by

$$\Phi(a)(x) = x.a, \qquad \forall a \in G, \forall x \in X.$$

The function $\Phi(a) : X \longrightarrow X$ is bijective, since it has an inverser, $\Phi(a^{-1})$. In fact, we have that

$$\Phi(a^{-1})(\Phi(a)(x)) = a^{-1}.(a.x) = (a^{-1}a).x = e.x = x, \qquad \forall x \in X.$$

We have that Φ is a homomorphism since

$$\Phi(ab).x = (ab).x = a.(b.x) = \Phi(a)(b.x) = \Phi(a)(\Phi(b)(x)),$$

for all $a, b \in G$ and all $x \in X$.

b). The action of r and s defines a homomorphism from the free group $F[\{r,s\}]$ to S_X, since the funtions given by

$$q(x) \mapsto \xi^{-2}q(\xi x) \quad \text{and} \quad q(x) \mapsto x^2 q(1/x)$$

are invertible with inverses

$$q(x) \mapsto \xi^{2}q(\xi^{-1} x) \quad \text{and} \quad q(x) \mapsto x^2 q(1/x).$$

We check that the relations are in the kernel of this map by

$$\begin{aligned} r^n.q(x) &= r^{n-1}.(r.q(x)) = r^{n-1}(\xi^{-2}q(\xi x)) = r^{n-2}.(x^{-4}q(\xi^2 x)) \\ &= \cdots = \xi^{-2n}q(\xi^n x) = q(x), \end{aligned}$$

and

$$s^2.q(x) = s.(s.q(x)) = s.(x^2q(1/x)) = x^2((1/x)^2q(x)) = q(x).$$

Furthermore, we have that

$$(rs).q(x) = r.(x^2q(1/x)) = \xi^{-2}((\xi x)^2 q(\xi^{-1}/x)) = \xi^2x^2q(\xi^{-1}/x)$$

and hence

$$(rs)^2.q(x) = \xi^2x^2(\xi^2(x^{-1}/x)^2q(\xi^{-1}/(\xi^{-1}/x)) = \xi^2x^2\xi^{-2}x^{-2}q(x) = q(x).$$

Hence we have that the kernel of the homomorphism $F[\{r,s\}] \longrightarrow S_X$ contains the relations and thus we have a well defined homomorphism

$$D_{2n} \longrightarrow S_X$$

accoding to problem 3.

---

(3) a) Show that if $\Phi : G \longrightarrow H$ is a group homomorphism and if K is a normal subgroup of G contained in $\ker \Phi$, then Φ factors through the natural quotient homomorphism $\Psi : G \longrightarrow G/K$, which sends an element of G to the coset containing it. **(2)**

b) Use the result in a) to show that the sign homomorphism, $\operatorname{sgn} : S_4 \longrightarrow \{\pm 1\}$, factors via $S_4 \longrightarrow S_3$. (*Hint*: use $K = \{\mathrm{Id}, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$.) **(4)**

SOLUTION

a). Define the map $\Psi : G/K \longrightarrow H$ by $\Psi(aK) = \Phi(a)$. This is well-defined since $aK = bK$ is equivalent to $b^{-1}a \in K$, which implies that $\Phi(b^{-1}a) = e_H$ since $K \subseteq \ker \Phi$. Hence we get that $\Phi(b)^{-1}\Phi(a) = e_H$, i.e., $\Phi(b) = \Phi(a)$.

It is a homomorphism since

$$\Psi(aK * bK) = \Psi(abK) = \Phi(ab) = \Phi(a)\Phi(b) = \Psi(aK)\Psi(bK),$$

for all $aK, bK \in G/K$. Now we have that $Phi(a) = \Psi(aK) = \Psi(\Xi(a)$, where $\Xi : G \longrightarrow G/K$ is the homomorphism given by $\Xi(a) = aK$, for $a \in G$. This means that $\Phi = \Psi \circ \Xi$ and Φ factors through Ξ

b). Let $K = \{\mathrm{Id}, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$. Since K consists of all permutations of type $[1^4] = 1+1+1+1$ and $[2^2] = 2+2$ it is closed under conjugation. Furthermore, it is closed under composition and under taking inverses sice they all satisfy $\sigma^2 = \mathrm{Id}$ and we have that

$$(1\,2)(3\,4) \circ (1\,3)(2\,4) = (1\,4)(2\,3),$$
$$(1\,2)(3\,4) \circ (1\,4)(2\,3) = (1\,3)(2\,4)$$

and

$$(1\,3)(2\,4) \circ (1\,4)(2\,3) = (1\,2)(3\,4).$$

Thus K is a normal subgroup and since it consists of only even permutations, it is contained in the kernel A_4 of the sign-homomorphism. Thus by the previous problem, sgn factors through $S_4 \longrightarrow S_4/K$. Now S_4/K is a group of order $|S_4|/|K| = 24/4 = 6$. Since S_4 does not contain any element of order greater than 4, the factor group S_4/K cannot do that either. Hence S_4/K is not cyclic and the only possibility is that $S_4/K \cong S_3$ since there is only one non-cyclic group of order 6 up to isomorphism. We have thus concluded that the sign homomorphism factors through S_3:

$$S_4 \longrightarrow S_4/K \cong S_3 \longrightarrow \{\pm 1\}.$$