FACES OF BIRKHOFF POLYTOPES

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ABSTRACT. The Birkhoff polytope $\mathcal{B}_n$ is the convex hull of all $(n \times n)$ permutation matrices, *i.e.*, matrices where precisely one entry in each row and column is one, and zeros at all other places. This is a widely studied polytope with various applications throughout mathematics.

In this paper we study combinatorial types $\mathcal{L}$ of faces of a Birkhoff polytope. The *Birkhoff dimension* $\mathrm{bd}(\mathcal{L})$ of $\mathcal{L}$ is the smallest n such that $\mathcal{B}_n$ has a face with combinatorial type $\mathcal{L}$.

By a result of Billera and Sarangarajan, a combinatorial type $\mathcal{L}$ of a d-dimensional face appears in some $\mathcal{B}_k$ for $k \leq 2d$, so $\mathrm{bd}(\mathcal{L}) \leq 2d$. We will characterize those types with $\mathrm{bd}(\mathcal{L}) \geq 2d-3$, and we prove that any type with $\mathrm{bd}(\mathcal{L}) \geq d$ is either a product or a wedge over some lower dimensional face. Further, we computationally classify all d-dimensional combinatorial types for $2 \leq d \leq 8$.

1. Introduction

The Birkhoff polytope $\mathcal{B}_n$ is the convex hull of all $(n \times n)$ permutation matrices, *i.e.*, matrices that have precisely one 1 in each row and column, and zeros at all other places. Equally, $\mathcal{B}_n$ is the set of all doubly stochastic $(n \times n)$-matrices, *i.e.*, non-negative matrices whose rows and columns all sum to 1, or the perfect matching polytope of the complete bipartite graph $K_{n,n}$. The Birkhoff polytope $\mathcal{B}_n$ has dimension $(n-1)^2$ with $n!$ vertices and n^2 facets. The Birkhoff-von Neumann Theorem shows that $\mathcal{B}_n$ can be realized as the intersection of the positive orthant with a family of hyperplanes.

Birkhoff polytopes are a widely studied class of polytopes [6, 7, 9–13, 16, 17, 19] with many applications in different areas of mathematics, *e.g.*, enumerative combinatorics [1, 38], optimization [2, 27, 39], statistics [26, 37], or representation theory [8, 34]. Yet, despite all these efforts, quite fundamental questions about the combinatorial and geometric structure of this polytope, and its algorithmic treatment, are still open. In particular, we know little about numbers of faces apart from those of facets, vertices, and edges.

In this paper, we study combinatorial types of faces of Birkhoff polytope. The *combinatorial type* of a face F of some Birkhoff polytope is given by its face lattice $\mathcal{L}$. For such a combinatorial type we can define the *Birkhoff dimension* $\mathrm{bd}(\mathcal{L})$ of $\mathcal{L}$ as the minimal n such that $\mathcal{B}_n$ has a face combinatorially equivalent to $\mathcal{L}$.

By a result of Billera and Sarangarajan [7] any combinatorial type of a d-dimensional face of $\mathcal{B}_n$ already appears in $\mathcal{B}_{2d}$, so $\mathrm{bd}(\mathcal{L}) \leq 2d$. Here, we characterize combinatorial types of d-dimensional faces with $\mathrm{bd}(\mathcal{L}) \geq 2d-3$. More precisely, we show in Theorem 5.12 that the only combinatorial type $\mathcal{L}$ with $\mathrm{bd}(\mathcal{L}) = 2d$ is the d-cube (Proposition 5.9). If $\mathrm{bd}(\mathcal{L}) = 2d-1$, then $\mathcal{L}$ must by a product of a cube and a triangle (Proposition 5.10). $\mathrm{bd}(\mathcal{L}) = 2d-2$

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allows three new types, a pyramid over a cube, the product of a cube with a pyramid over a cube, and the product of two triangles with a cube (Proposition 5.11). Finally, faces with $\mathrm{bd}(\mathcal{L}) = 2d - 3$ are either products or certain Cayley polytopes of products of lower dimensional faces, the joined products and reduced joined products defined in Section 5.4 (Theorem 5.16).

More generally, we show in Section 5.1 that any combinatorial type $\mathcal{L}$ of a d-dimensional face with $\mathrm{bd}(\mathcal{L}) \geq d$ is either a product of two lower dimensional faces, or a wedge of a lower dimensional face over one of its faces (Corollary 5.2). We further characterize combinatorial types of faces F of some Birkhoff polytope for which the pyramid over F is again a face of some Birkhoff polytope.

Finally, we enumerate all combinatorial types of d-dimensional faces of some $\mathcal{B}_n$ for $2 \leq d \leq 8$. This is done with an algorithm that classifies face graphs corresponding to combinatorially different faces of $\mathcal{B}_n$. The algorithm has been implemented as an extension to the software system `polymake` [28,36] for polyhedral geometry (Section 6). The computed data in `polymake` format can be found at [35].

Following work of Billera and Sarangarajan [7] we use elementary bipartite graphs (*face graphs*) to represent combinatorial types of faces. A graph is *elementary* if every edge is contained in some perfect matching in the graph. A perfect matching in a bipartite graph with n nodes in each layer naturally defines an $(n \times n)$-matrix with entries in $\{0, 1\}$, which gives the correspondence to a face of $\mathcal{B}_n$. The correspondence of faces and graphs is explained in Section 2.2. We use the language of face graphs in Section 3 and Section 4 to study the structure of these graphs and the corresponding faces.

Previously, Brualdi and Gibson have done an extensive study of faces of Birkhoff polytopes in a series of papers [10–13]. They used 0/1-matrices to represent types of faces, which naturally correspond to elementary bipartite graphs by placing edges at all non-zero entries. They studied combinatorial types of faces with few vertices, the diameter of $\mathcal{B}_n$, and some constructions for new faces from given ones. We review some of their results in Section 4, as we need them for our constructions in Section 5.

A fair amount of work also has gone into the computation of the Ehrhart polynomial or the volume of the Birkhoff polytope. Until recently, only low dimensional cases were known [6,19] using a computational approach. In 2009, Canfield and McKay [16] obtained an asymptotic formula for the volume, and in the same year De Loera et al. [17] gave an exact formula by computing the Ehrhart polynomial.

Birkhoff polytopes are a special case of the much more general concept of a permutation polytope. These are polytopes obtained as the convex hull of all permutation matrices corresponding to some subgroup G of the the full permutation group S_n. So the Birkhoff polytope is the permutation polytope of S_n. Permutation polytopes have been introduced by Guralnick and Perkinson in [29]. They studied these objects from a group theoretic view point and provided formulas for the dimension and the diameter. A systematic study of combinatorial properties of general permutation polytopes and a computational classification of d-dimensional permutation polytopes and d-dimensional faces of some higher dimensional permutation polytope for $d \leq 4$ can be found in [4].

Several subpolytopes of the Birkhoff polytope have been shown to have an interesting structure and some beautiful properties. Here, in particular the polytope of *even* permutation matrices attracted much attention [14, 25, 31], but also many other classes of groups have been considered [3, 5, 20, 40].

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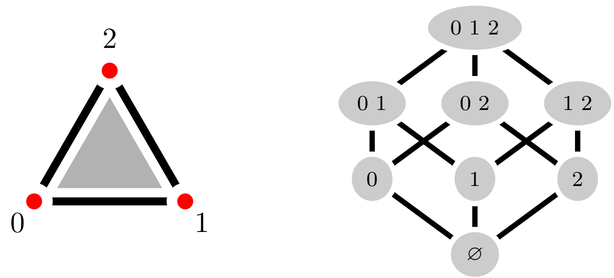


Figure 2.1. A triangle and its face lattice.

by a postdoc position in the Emmy-Noether project HA 4383/1 and the Priority Program SPP 1489 of the German Research Society (DFG). The computation of low dimensional faces has been done with an extension to the software system `polymake` [28].

2. Background and Basic Definitions

2.1. **Polytopes.** A *polytope* $P \subseteq \mathbb{R}^d$ is the convex hull $P = \mathsf{conv}(v_1, \ldots, v_k)$ of a finite set of points $v_1, \ldots, v_k \in \mathbb{R}^d$. Dually, any polytope can be written as the bounded intersection of a finite number of affine half-spaces in the form $P := \{x \mid Ax \leq b\}$. We repeat some notions relevant for polytopes. For a thorough discussion and proofs we refer to [41].

A (proper) *face* F of a polytope P is the intersection of P with an affine hyperplane H such that P is completely contained in one of the closed half-spaces defined by H. (The intersection may be empty.) We also call the empty set and the polytope P a face of P. Any face F is itself a polytope. The dimension of a polytope $P \subseteq \mathbb{R}^d$ is the dimension of the minimal affine space containing it. It is *full-dimensional* if its dimension is d.

0-dimensional faces of P are called *vertices*, 1-dimensional faces are *edges*. Proper faces of maximal dimension are called *facets*. P is the convex hull of its vertices, and the vertices of any face are a subset of the vertices of P. Thus, a polytope has only a finite number of faces. Let f_i be the number of i-dimensional faces of P, $0 \leq i \leq \dim P - 1$. The f-vector of a d-dimensional polytope P is the non-negative integral vector $f(P) := (f_0, \ldots, f_{d-1})$.

Inclusion of sets defines a partial order on the faces of a polytope. The *face lattice* or *combinatorial type* $\mathcal{L}(P)$ of a polytope P is the partially ordered set of all faces of P (including the empty face and P itself). This defines a Eulerian lattice. See Figure 2.1 for an example. It contains all combinatorial information of the polytope. Two polytopes P, P' are *combinatorially isomorphic* or *have the same combinatorial type* if their face lattices are isomorphic as posets. For any given Eulerian lattice $\mathcal{L}$ we call a subset $P \subset \mathbb{R}^d$ a *geometric realization* of $\mathcal{L}$ if P is a polytope with a face lattice isomorphic to $\mathcal{L}$. Note, that not all Eulerian lattices are a face lattice of a polytope.

An *r-dimensional simplex* (or r-simplex) is the convex hull of $r+1$ affinely independent points in $\mathbb{R}^d$. A polytope is called *simplicial* if all facets are simplices. It is *simple* if the dual is simplicial. Equally, a d-dimensional polytope P is simple if each vertex is incident to precisely d edges. The *d-dimensional $0/1$-cube* C^d is the convex hull of all d-dimensional $0/1$-vectors. This is a simple d-polytope with 2^d vertices and $2d$ facets. More generally, we denote by a *d-cube* any d-dimensional polytope that is combinatorially isomorphic to the $0/1$-cube (it need not be full dimensional).

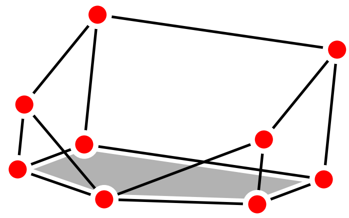


Figure 2.2. The wedge over a vertex of a pentagon.

Let $P_1 \subset \mathbb{R}^{d_1}$ and $P_2 \subset \mathbb{R}^{d_2}$ be two (geometrically realized) polytopes with vertex sets $\mathbf{V}(P_1) = \{v_1, \ldots, v_k\}$ and $\mathbf{V}(P_2) = \{w_1, \ldots, w_l\}$. With $\mathbf{0}^{(d)}$ we denote the d-dimensional zero vector.

The *(geometric) product* of P_1 and P_2 is the polytope

$$P_1 \times P_2 \;:=\; \mathsf{conv}\left((v_i, w_j) \in \mathbb{R}^{d_1+d_2} \mid 1 \le i \le k, 1 \le j \le l \right).$$

This is the same as the set of all points (v, w) for $v \in P_1$ and $w \in P_2$. The *(geometric) join* of P_1 and P_2 is the polytope

$$P_1 \star P_2 \;:=\; \mathsf{conv}\Big(P_1 \times \{\mathbf{0}^{(d_2)}\} \times \{0\} \cup \{\mathbf{0}^{(d_1)}\} \times P_2 \times \{1\} \Big) \;\subseteq\; \mathbb{R}^{d_1+d_2+1}.$$

More generally, we say, that a polytope P is a *product* or *join* of two polytopes P_1 and P_2, if P is combinatorially isomorphic to the geometric product or geometric join of some realizations of the face lattices of P_1, or P_2.

If F is a face of a polytope $P := \{x \mid Ax \le b\} \subseteq \mathbb{R}^d$ and $\langle c, x \rangle \le d$ a linear functional defining F, then the *wedge* $\mathsf{wedge}_F(P)$ of P over F is defined to be the polytope

$$\mathsf{wedge}_F(P) \;:=\; \left\{ (x, x_0) \in \mathbb{R}^{d+1} \mid Ax \le b,\, 0 \le x_0 \le d - \langle c, x \rangle \right\}. \tag{2.1}$$

See Figure 2.2 for an example. Again, we say more generally that P is a *wedge* of a polytope Q over some face F of Q if P is combinatorially equivalent to $\mathsf{wedge}_F(Q)$.

We also extend these notions to combinatorial types, *i.e.*, we say that a combinatorial type $\mathcal{L}$ (or face lattice) of a polytope P is a *cube*, *simplex*, *product*, *join*, or *wedge*, if some geometric realization (and, hence, also any other) of $\mathcal{L}$ is.

With $\mathcal{N}(v)$ for a node v of a graph G we denote the *neighborhood* of v, *i.e.*, the set of all nodes in G that are connected to v by an edge. If M is a set of nodes in G, then we denote by $\mathcal{CN}(M)$ the *set of common neighbors* of all nodes in M, *i.e.*, the set

$$\mathcal{CN}(M) \;:=\; \bigcap_{v \in M} \mathcal{N}(v)\,.$$

See Figure 2.3 for an example.

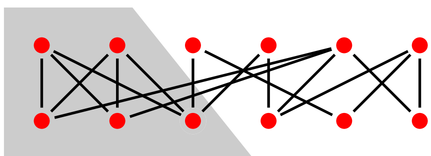


Figure 2.3. The common neighbors of the first three nodes in the lower layer are the first two nodes in the upper layer.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

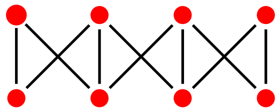


Figure 2.4. A face of $\mathcal{B}_3$ and its graph. The upper layer represents the rows, the lower layer the columns of the matrix. An edge of the graph represents a 1 in the matrix at the position corresponding to its end points.

2.2. **The Birkhoff polytope.** Let S_n be the group of permutations on n elements. To any element $\sigma \in S_n$ we can associate a 0/1-matrix $M(\sigma) \in \mathbb{R}^{n\times n}$ that has a 1 at position (i, j) if and only if $\sigma(i) = j$. The *n-th Birkhoff polytope* is

$$\mathcal{B}_n \ := \ \mathsf{conv}\,(\,M(\sigma) \mid \sigma \in S_n) \ \subseteq \ \mathbb{R}^{n\times n}\,.$$

The Birkhoff-von Neumann Theorem shows that $\mathcal{B}_n$ can equally be characterized as the set of all non-negative $(n \times n)$-matrices whose rows and columns all sum to 1. Equivalently, the facets of $\mathcal{B}_n$ are precisely defined by the inequalities $x_{ij} \geq 0$ for $1 \leq i, j \leq n$. It has dimension $(n-1)^2$ with n^2 facets and $n!$ vertices.
More generally, we associate a 0/1-matrix $M(\Sigma) \in \mathbb{R}^{n\times n}$ to any subset $\Sigma \subseteq S_n$ in the following way. $M(\Sigma)$ has a 1 at position (i, j) if there is some $\tau \in \Sigma$ with $\tau(i) = j$, and 0 otherwise. If $\Sigma = \{\sigma\}$ for some $\sigma \in S_n$, then $M(\Sigma) = M(\sigma)$.
We can view $M(\Sigma)$ as a dual vector in $(\mathbb{R}^{n\times n})^*$. The functional $M(\Sigma)$ satisfies

$$\langle M(\Sigma), x\rangle \ \leq \ n \ \text{ for all } x \in \mathcal{B}_n\,.$$

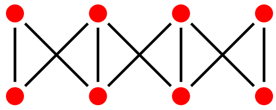
Any $x = M(\sigma)$ for a $\sigma \in \Sigma$ satisfies this with equality, so this inequality defines a proper non-empty face

$$\mathsf{F}(\Sigma) \ := \ \{\,M(\sigma) \mid \langle M(\Sigma), M(\sigma)\rangle = n\,\}\ .$$

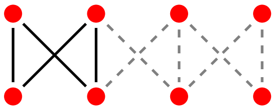
of the polytope $\mathcal{B}_n$, and all $\sigma \in \Sigma$ are vertices of that face. However, there may be more. Namely, any permutation τ such that for any $i \in [n]$ there is $\sigma \in \Sigma$ with $\tau(i) = \sigma(i)$ is also a vertex of $\mathsf{F}(\Sigma)$. The well-known fact that any face is defined by a subset of the facet inequalities implies the following proposition.

Proposition 2.1. *Any face F of $\mathcal{B}_n$ is of the type $\mathsf{F}(\Sigma)$ for some $\Sigma \subseteq S_n$, and $\mathsf{F}(\Sigma)$ is the smallest face containing all vertices corresponding to elements of Σ.*

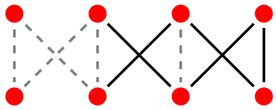
Different subsets of S_n may define the same face of $\mathcal{B}_n$, so this correspondence is not a bijection. For example, $\mathsf{F}(\Sigma)$ is the same square in $\mathcal{B}_4$ for either of the sets $\Sigma = \{(), (1\ 2)(3\ 4)\} \subset S_4$ and $\Sigma = \{(1\ 2), (3\ 4)\} \subset S_n$ (and the vertices of the square correspond to the union of those two sets).
For the following considerations there is a different representation of faces that is easier to deal with. For any subset $\Sigma \subseteq S_n$ we associate a bipartite graph $\Gamma(\Sigma)$ with n nodes in each color class to the matrix $M(\Sigma)$. Let $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$ be two disjoint vertex sets and draw an edge between the nodes u_i and v_j if and only if there is $\sigma \in \Sigma$ with $\sigma(i) = j$. This gives a bipartite graph with two color classes U and V of equal size n. In the following, we call U the *upper layer* and V the *lower layer*. Figure 2.4 shows an example. In this example, Σ can be chosen to contain the identity permutation and the transpositions $(1\ 2)$, $(2\ 3)$ and $(3\ 4)$. The face $\mathsf{F}(\Sigma)$ also contains the vertex corresponding to the permutation $(1\ 2)(3\ 4)$. Clearly, the bipartite graph is just a different representation



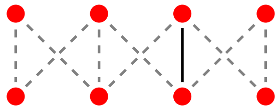
(a) A face graph G with three ears P_1, P_2, and P_3



(b) The ear P_1



(c) The ear P_2



(d) The ear P_3

Figure 2.5. An ear decomposition of an elementary graph

of the matrix. We can recover the matrix by putting a 1 at each position (i, j) where node i of the upper layer is connected to node j of the lower layer, and 0 everywhere else.

Any vertex of the face $\mathsf{F}(\Sigma)$ corresponds to a perfect matching in the graph $\Gamma(\Sigma)$, and any perfect matching in the graph defines a vertex. Conversely, any bipartite graph with the property that every edge is contained in a perfect matching defines a face of $\mathcal{B}_n$ as the convex hull of the permutations defined by its perfect matchings. In the following, we will use the term *face graph* for bipartite graphs such that each edge is contained in some perfect matching of the graph.

Note, that in the literature graphs in which every edge is in some perfect matching are called *elementary*. So a face graph is a bipartite elementary graph. Elementary graphs are well-studied objects, see, *e.g.*, the work of Lovász [32], Lovász and Plummer [33], Brualdi and Shader [15], and the extensive work of de Carvalho, Lucchesi and Murty [18, 21–24]. In the special case of bipartite graphs, being elementary implies that both layers have the same number of nodes. An important property of elementary graphs is the existence of an ear decomposition, which we will explain now.

Definition 2.2. An *ear decomposition* of an elementary graph G is a decomposition of the graph into edge disjoint paths and cycles $P_1, P_2, \ldots, P_r$ subject to the following two conditions:

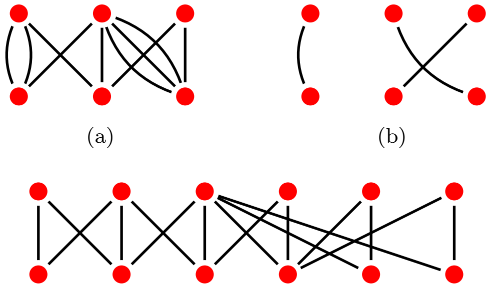
(1) P_1 is a cycle.
(2) If P_i, $1 \leq i \leq r$ is a path, then its endpoints lie in different layers of G. These are the only two points that P_i has in common with the graph $P_1 \cup P_2 \cup \ldots \cup P_{i-1}$.
(3) If P_i, $1 \leq i \leq r$ is a cycle, then P_i is disjoint from $P_1 \cup P_2 \cup \ldots \cup P_{i-1}$.

See Figure 2.5 for an example. The following result can, *e.g.*, be found in the book of Lovász and Plummer [33, Thm. 4.1.6].

Theorem 2.3. *A bipartite graph G is elementary if and only if all its connected components have an ear decomposition.* □

By a simple counting argument one can show that the number of ears is independent of the chosen ear decomposition. If n is the number of vertices in each layer, m the number of edges, and k the number of connected components, then the graph has $r = m - 2n + k + 1$ ears. The ear decomposition also guarantees that an elementary graph is 2-connected and any node has degree at least 2.

Let $\mathsf{FG}(n)$ be the set of face graphs where each layer has n vertices. By the above there is a bijection between faces of $\mathcal{B}_n$ and elements of $\mathsf{FG}(n)$. Let $\Gamma(F)$ be the face graph



(a) (b) (c)

Figure 2.6. A face multi-graph $\overline{G}$, a perfect matching in $\overline{G}$, and the resolution $\mathsf{res}(\overline{G})$ of G.

corresponding to the face F. For faces E, F of $\mathcal{B}_n$ the face E is a face of F if and only if $\Gamma(E)$ is a subgraph of $\Gamma(F)$. Hence, we can read off the face lattice of a face F from its graph $\Gamma(F)$. Two graphs that define isomorphic lattices are said to have the same *combinatorial type.*

For a face graph G with upper layer U and any set $S \subseteq U$ we have $|\mathcal{N}(S)| > |S|$ unless $S \cup \mathcal{N}(S)$ is the vertex set of a connected component of G (otherwise, any edge with one end in $\mathcal{N}(S)$ and one not in S could never be part of a perfect matching). Hence, if G is a face graph then any graph obtained by adding edges without reducing the number of connected components is again a face graph.

Edges of the face correspond to unions of two perfect matchings M_1, M_2 that do not imply any other perfect matchings. Hence, for an edge, $M_1 \cup M_2$ is a disjoint union of edges and a single cycle.

To study face graphs and the faces of $\mathcal{B}_n$ they define it is sometimes convenient to consider a more general representation of a face as a graph. A multi-graph $\overline{G}$ is a graph where more than one edge between two nodes is allowed. As for simple graphs, a matching in a multi-graph is a selection of edges such that no vertex is incident to more than one edge. It is perfect if every node is incident to precisely one edge. Again, we can define the associated lattice of face multi-graphs and their combinatorial types.

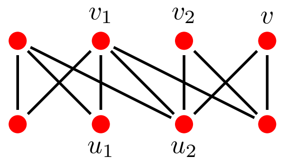
In a face graph we can replace any edge with a path of length 3 without changing the number of perfect matchings and their inclusion relation. For a face multi-graph $\overline{G}$ we define its *resolution* $\mathsf{res}(\overline{G})$ to be the graph obtained from $\overline{G}$ by replacing all but one edge between any pair of nodes by a path of length 3. See Figure 2.6 for an example.

We can also define a converse operation. Let $\Gamma(F)$ be a face graph. For a vertex v of degree 2 we introduce the *reduction* $\mathsf{red}(\Gamma(F), v)$ at a vertex v. Let F be a face graph with a vertex v of degree 2 and neighbors u_1, u_2. The reduction $\mathsf{red}(\Gamma(F), v)$ of F at v is the graph obtained from $\Gamma(F)$ by contracting v. This graph may have double edges. See Figure 2.7 for an illustration. This construction already appears in the paper of Billera and Sarangarajan [7]. We summarize the properties of $\Gamma(F)$ and a face multi-graph $\overline{G}$.

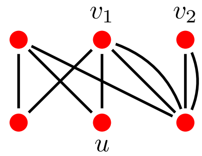
Proposition 2.4. *Let F be a face of some Birkhoff polytope, and $\overline{G}$ a face multi graph corresponding to F. Then*

(1) $\Gamma(F)$ and its reduction $\mathsf{red}(\Gamma(F))$ have the same combinatorial type,

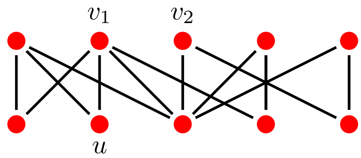
(2) $\overline{G}$ and its resolution $\mathsf{res}(\overline{G})$ have the same combinatorial type. □



(a) The graph G



(b) The graph $\mathsf{red}(G, v)$



(c) The graph $\mathsf{res}(\mathsf{red}(G, v))$

Figure 2.7. Contracting a vertex of degree 2

In the next sections we want to study combinatorial types of faces of $\mathcal{B}_n$ by studying their face graphs. The following proposition tells us that we can mostly restrict our attention to connected face graphs.

Proposition 2.5. *Let G be a face graph with connected components $G_1, \ldots, G_k$, $k \geq 2$. Then $G_1, \ldots, G_k$ are face graphs and the face corresponding to G is the product of the faces corresponding to $G_1, \ldots, G_k$.*

Proof. Any perfect matching in G induces a perfect matching in G_i, $1 \leq i \leq k$. Hence, any edge in G_i is contained in a perfect matching of G_i, so all G_i are face graphs. Perfect matchings correspond to vertices, and any combination of perfect matchings in the G_i defines a perfect matching of G. Hence, the face of G is a product. □

In particular, prisms over faces are again faces, and their face graph is obtained by adding a disjoint cycle of length 4.
The converse statement, *i.e.*, that the face graph of a face that combinatorially is a product of two other faces is always disconnected, follows, *e.g.*, from [11, Cor. 4.7], where they prove that the (vertex-edge) graph of two faces is isomorphic if and only if the two face graphs have the same number of components and there is a correspondence between the components such that each pair has isomorphic graphs.
We can read off the dimension of $\mathsf{F}(\Sigma)$ for $\Sigma \subseteq S_n$ from the graph $\Gamma(\Sigma)$. Assume first that the face graph is connected with n nodes in each layer and m edges. Then the Birkhoff-von Neumann Theorem implies that the dimension d is at most $m - 2n + 1$. On the other hand, the graph has an ear decomposition with $m - 2n + 2$ ears, so we have at least $m - 2n + 2$ linearly independent vertices in the face. So $d = m - 2n + 1$ for a connected face graph. Using that disconnected graphs define products we obtain

$$\dim \mathsf{F}(\Sigma) = m - 2n + k\,, \tag{2.2}$$

where k is the number of connected components of the graph. The following theorem of Billera and Sarangarajan tells us that we can restrict the search for combinatorial types of d-dimensional faces to $\mathcal{B}_n$ for $n \leq 2d$. We repeat the simple proof, as it is quite instructive.

Theorem 2.6 (Billera and Sarangarajan [7])**.** *Any d-dimensional combinatorial type of face already appears in $\mathcal{B}_{2d}$.*

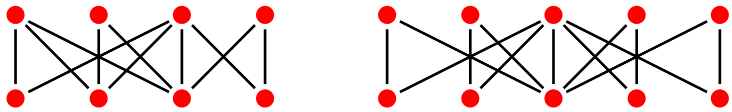


Figure 3.1. Two irreducible graphs both defining a 4-simplex.

Proof. We assume first that the face graph G is connected. Then G has $m = 2n + (d-1)$ edges. Let k_2 be the number of nodes of degree 2. A node in G has degree at least 2, so

$$k_2 \;\geq\; 2n - 2(d-1)\,.$$

We now successively contract all nodes of degree 2 using the above reduction. We obtain a face multi-graph G' on $2n' = 2n - k_2 \;\leq\; 2(d-1)$ nodes, *i.e.*, at most $d-1$ on each layer. Note that one reduction step can remove more than one node of degree 2. Let m' be the number of edges of G'. G' defines a face combinatorially equivalent to the one of G, in particular, its dimension is $d = m' - 2n' + 1$. The graph G' may have multiple edges. Let e' be the minimal number of edges we have to remove to obtain a simple graph $\overline{G}$. Then $\overline{G}$ is connected and a face graph corresponding to a face of dimension

$$0 \;\leq\; \overline{d} \;=\; m' - 2n' + 1 - e' \;=\; d - e'\,.$$

Thus, $e' \leq d$, and we can resolve each multiple edge to obtain a face graph $\tilde{G}$ with at most $2(d-1) + 2d \leq 2(2d-1) \leq 4d$ nodes. Hence, $\tilde{G}$ defines a face of $\mathcal{B}_n$ that is combinatorially isomorphic to the one of G.

The statement for disconnected graphs follows using induction by replacing the graph in each component with the above procedure. □

3. Irreducibility

In general, a combinatorial type of a face can occur many times as a geometrically realized face of $\mathcal{B}_n$. Hence, there are many different possibilities to represent a combinatorial type of a face as an face graph. Brualdi and Gibson [13, Conj. 1] conjecture that any two combinatorially isomorphic faces are affinely equivalent, but as far as we know this is still open.

Let G be a face graph. In the following, we want to examine some version of *minimality* for such a representation. This will, however, not lead to a unique "standard" representation. We say that a node v in G is *reducible*, if v has degree 2 in G and the common neighborhood of the two vertices adjacent to v only contains the node v. v is called *irreducible* otherwise. A face graph G is called *irreducible*, if all its nodes are irreducible, and *reducible* otherwise. An irreducible representation of a certain d-face of a Birkhoff polytope need not be unique. Figure 3.1 shows two irreducible representations of the 4-simplex on a different number of nodes. See Figure 3.2 for an example of a reducible node. By Proposition 2.4 the face graphs G and $G' := \mathsf{red}(G, v)$, for any node v of G, define the same combinatorial type. Hence, we can mostly restrict our considerations to irreducible face graphs.

Lemma 3.1. *Let G be a face graph, $v \in G$ an irreducible node of degree 2 in G and w_1, w_2 its neighbors. Let*

$$N := (\mathcal{N}(w_1) \cap \mathcal{N}(w_2)) \setminus \{v\}\,.$$

Then either the graph induced by v, w_1, w_2 and the nodes in N is a connected component of G (and then necessarily the set N contains a single node u) or all points $u \in N$ and at least one of w_1, w_2 have degree ≥ 3.

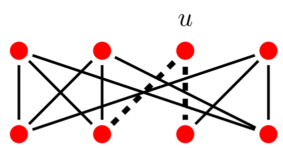


Figure 3.2. A reducible face graph: The node u incident to the two thick dashed edges is reducible.

Proof. We prove this by contradiction. We are done, if v, w_1, w_2 and the nodes in N form a connected component. Hence assume this is not the case.
Assume first that w_1, w_2 both have degree 2. Then there is some node $u \in N$ that has a third neighbor x different from w_1, w_2. However, the edge (u, x) must be contained in a perfect matching M of the graph. This perfect matching cannot use the edges (u, w_1) and (u, w_2). As both nodes w_1, w_2 have degree 2, M must use the remaining edge on both nodes. But these both contain v, so M is not a perfect matching.
So one of w_1, w_2 must have degree ≥ 3. Assume this is w_1, and let u, x be the two neighbors of w_1 different from v. If u would have degree 2, then u and v are contained in the edges $e_i := (u, w_i)$, $f_i := (v, w_i)$, $i = 1, 2$, and no other. Hence, any perfect matching M must choose either e_1 or e_2, and, correspondingly, f_2 or f_1. In either case w_1 is covered, hence, the edge (w_1, x) can never be chosen, so G is not a face graph. □

If G is an irreducible face graph then we say that a node v is *minimal* if the degree of v is 2. For a minimal node v the set

$$\mathcal{P}(v) := \mathcal{CN}(\mathcal{N}(v)) \setminus \{v\}$$

is the set of *partners* of the node v. This is the same as the set of nodes connected to v via two different paths of length 2. A node $x \in \mathcal{P}(v)$ is called a *partner* of v.
Note, that any partners of a node always lie in the same layer as the node itself. Lemma 3.1 guarantees that any minimal node in an irreducible face graph has at least one partner. We use the term *partner* more generally for any node x that is a partner of some other v, without reference to the node v. In particular, x can be partner of several different nodes in G. However, the next corollary bounds this number from above.

Proposition 3.2. *Let G be a face graph. Any partner x in G of degree k has at most $k-1$ nodes it is partner for.*
Moreover, if x is a partner for precisely $k-1$ nodes, then these $k-1$ nodes and x are the upper or lower layer of a connected component in G.

Proof. If v is a node that has x as its partner, then in any perfect matching M, v uses up one of the nodes adjacent to x for the edge covering v. Now also x needs to be covered, hence there can be at most $k-1$ nodes choosing x as partner.
If x is a partner for precisely $k-1$ nodes $v_1, \ldots, v_{k-1}$, then in any perfect matching in G, all but one node in the neighborhood of x is covered by an edge that has one endpoint among the v_i, $1 \leq i \leq k-1$ or x. But also x needs to be covered, hence, there cannot be another edge that ends in a node in the neighborhood of x. □

Corollary 3.3. *A connected irreducible face graph of dimension d with n nodes has at most d nodes of degree 2 in each layer, if $n = d + 1$, and at most $d - 1$ otherwise.*

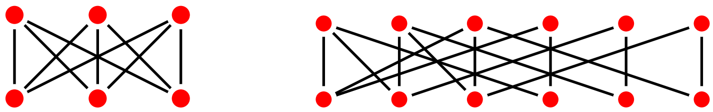


Figure 3.3. Face graphs with nodes of maximal degree

Proof. Let k_2 be the number of nodes of degree 2 (minimal nodes) in the upper layer. If all minimal nodes have the same partner, then, by the previous proposition, the graph has $k_2 + 1$ nodes, and $2k_2 + 1 + k_2 = 3k_2 + 1$ edges. Hence, $d = 3k_2 + 1 - 2k_2 - 2 + 1 = k_2$. Otherwise, we have at least two partners in the upper layer, and the previous proposition implies $2n + k_2 \leq d + 2n - 1$, *i.e.*, $k_2 \leq d - 1$. □

Corollary 3.4. *A connected irreducible d-dimensional face graph G has at most $2d - 2$ nodes.*

Proof. By the previous corollary the graph has at most d nodes if only one node has degree greater than 2. Otherwise, we have at most $d - 1$ nodes of degree 2, and for each of those we need a partner of degree at least 3. This leaves us with $2d - 2$ nodes using up all $2n + d - 1$ edges. □

Proposition 3.5. *Let G be an connected irreducible face graph of dimension d on n nodes. Then the maximum degree of a node in G is $2d - n + 1$ if $n > d + 1$ and n otherwise.*

Proof. The bound for $n \leq d + 1$ is trivial. If G has dimension d then G has $d + 2n - 1$ edges, and any node has degree at least 2. Let k_2 be the number of nodes of degree 2 in the upper layer and δ the degree of a non-minimal node v. v can be partner for at most $\delta - 2$ nodes, as otherwise $n \leq d + 1$. Any minimal node has degree 2, and any other node at least degree 3. Hence,

$$\begin{aligned} \delta - 3 &\leq d + 2n - 1 - (2k_2 + 3(n - k_2)) = d - n - 1 + k_2 \\ &\leq d - n - 1 + d - 1 = 2d - n - 2\,. \end{aligned}$$

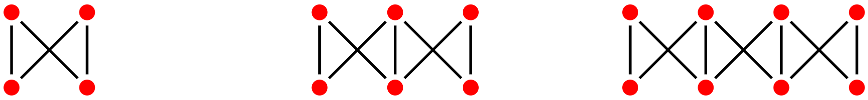
This implies the bound. □

The bound is best possible, see Figure 3.3.

Corollary 3.6. *Let G be a connected irreducible face graph with $n \geq 4$ nodes on each of its layers. Then G has at least $2n + \lceil \frac{n}{2} \rceil$ edges.*

Proof. Any node has degree at least 2. By Lemma 3.1 we have to find a partner of higher degree for each node of degree 2. On the other hand, Proposition 3.2 limits the number of minimal nodes a node can be partner for. We distinguish two cases:
If there is a node u in the graph that is a partner for all minimal nodes, then necessarily $\deg(u) = n$, hence the graph has $3n - 2 \geq 2n + \lceil \frac{n}{2} \rceil$ edges.
Otherwise, there are k_2 minimal nodes and $p \geq 2$ partners in the graph. We consider the cases $k_2 \geq p$ and $k_2 < p$ separately.
In the first case we have $k_2 \geq p$, hence $p \leq \lfloor \frac{n}{2} \rfloor$. The p partners in the graph together must be adjacent to at least $2p + k_2$ edges, hence we have at least

$$2k_2 + 2p + k_2 + 3(n - k_2 - p) = 3n - p \geq 2n + \left\lceil \frac{n}{2} \right\rceil$$



(a) The graph of a segment (b) The graph of a triangle (c) The graph of a square pyramid

Figure 3.4. Graphs with a minimal number of edges.

edges in the graph. In the second case the number of edges is at least

$$2k_2 + 3p + 3(n - k_2 - p) = 3n - k_2 \geq 2n + \left\lceil \frac{n}{2} \right\rceil \qquad \square$$

Proposition 3.7. *An irreducible face graph with two nodes in each layer has four edges, an irreducible face graph with three nodes in each layer has at least seven edges.*

Proof. The first case is trivial (see Figure 3.4(a)). For the second case just observe that we need to have at least one node of degree ≥ 3. $\square$

The given bounds are tight, as the graphs in Figure 3.4 show.

Proposition 3.8. *Let G be an irreducible face graph on n nodes and v a node of degree k in G. Then at most $k-1$ neighbors of v can have degree 2.*

Proof. v must be connected to the partner of all nodes in its neighborhood that have degree 2. $\square$

4. The Structure of Faces of $\mathcal{B}_n$

Here we review some basic properties of facets and faces of Birkhoff polytopes that we need for our classifications in the following section.

There is a quite canonical way to split the set of vertices of a face of $\mathcal{B}_n$ into two non-empty subsets on parallel hyperplanes at distance one. Let G be a face graph with some edge e, M_e the set of all perfect matchings in G containing e, and $M_{\neg e}$ its complement. Clearly, both M_e and $M_{\neg e}$ define faces of $\mathsf{F}(G)$, and $G = M_e \cup M_{\neg e}$ (not disjoint). Geometrically, if e connects the nodes i and j, then all vertices of M_e satisfy $x_{ij} = 1$, while all others lie on the hyperplane $x_{ij} = 0$.

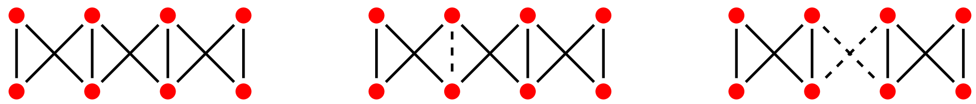
We start with some properties of facets of a face F of $\mathcal{B}_n$ with face graph $\Gamma(F)$. The face graph of a facet of F is a face subgraph of $\Gamma(F)$. We call a set C of edges in $\Gamma(F)$ *facet defining* if $\Gamma(F) - C$ is the face graph of a facet of F. Brualdi and Gibson [12, p. 204f] show that a facet defining set C is a (usually not perfect) matching in $\Gamma(F)$ and that any two different facet defining sets are disjoint. This leads to the following characterization of face subgraphs of facets.

Theorem 4.1 (Brualdi and Gibson [12, Cor. 2.11])**.** *Let G be a connected face graph and S a face subgraph of G. $\mathsf{F}(S)$ is a facet of $\mathsf{F}(G)$ if and only if either*

(1) S is connected and differs from G by a single edge, or

(2) S splits into disjoint face graphs $S_1, \ldots, S_k$ such that there are nodes $u_i, v_i \in S_i$ inducing a decomposition of G as

$$G \;=\; S_1 \cup \ldots S_k \cup \{(u_1, v_2), (u_2, v_3), \ldots, (u_k, v_1)\}\,.$$



(a) A face graph G

(b) A facet of the first type of G. The edges not in S are drawn with dashed lines.

(c) A facet of the second type of G. The edges not in S are drawn with dashed lines.

Figure 4.1. A face graph with two types of facets.

Note that in the second case u_i and v_i are necessarily on different layers of the graph. See Figure 4.1 for an illustration of the two types.
It follows from a Theorem of Hartfiel [30, Theorem ⋆] that any connected face graph G with $\dim(\mathsf{F}(G)) \geq 2$ is reducible if all facet defining sets in G have two or more edges. Geometrically, this implies the following lemma.

Lemma 4.2. *Let F be a face of $\mathcal{B}_n$. If F is not a product, then F has a facet that is not a product.*

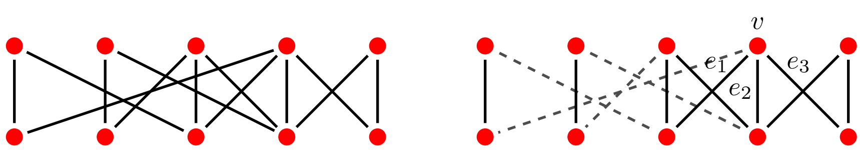
Proof. Let G be a connected irreducible graph representing F. By the above, G has a facet defining set C of size one. Hence $H := G - C$ is connected, as G is 2-connected, and the facet defined by H is not a product. □

Brualdi and Gibson in their papers also obtained some results about edges and 2-dimensional faces of $\mathcal{B}_n$.

Lemma 4.3 (Brualdi and Gibson [11, Lemma 3.3 and Lemma 4.2])**.** *Let G be a connected face graph and e_1, e_2, e_3 edges with a common node v.*

(1) There are perfect matchings M_1 and M_2 each containing one of the edges such that $\mathsf{F}(M_1 \cup M_2)$ is an edge in some $\mathcal{B}_n$.
(2) If there are perfect matchings M_1, M_2 with $e_i \in M_i$, $i = 1, 2$, such that $\mathsf{F}(M_1 \cup M_2)$ is an edge, then there is a perfect matching M_3 containing e_3 such that $\mathsf{F}(M_1 \cup M_2 \cup M_3)$ is a triangle in some $\mathcal{B}_n$.

See Figure 4.2 for an example. More generally, the union of any two perfect matchings in a face graph is the disjoint union of single edges and cycles. Hence, the minimal face containing a given pair of vertices is always a cube of some dimension.
Note, that Lemma 4.3 ensures for any two edges sharing a node the existence of two perfect matchings containing them that form an edge. Hence, any three edges with a common node define at least one triangle in the polytope. This implies that the only triangle free faces of $\mathcal{B}_n$ are cubes [13, Thm. 4.3].



(a) A face graph G

(b) A triangle in G based at v

Figure 4.2. A face graph and a triangle in that graph.

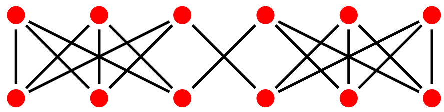


Figure 4.3. A 3-regular graph for a 7-dimensional face with $17 < 3(7-1)$ facets

In fact, any vertex of a face F of $\mathcal{B}_n$ is incident to at least one triangle, unless F is a cube. Theorem 4.4 of [11] furthermore tells us that the induced graph of the neighborhood of any vertex in the polytope graph has k components if and only if the polytope is a k-fold product. Note that one direction is trivial. If the face is a product, then the union of the perfect matchings of all neighbors of a vertex is already the graph of the face.

Brualdi and Gibson [13, Thm. 3.3] showed that a d-dimensional face $F := \mathsf{F}(G)$ corresponding to an irreducible face graph G has at most $3(d-1)$ facets, which is linear in d. Further, if F has exactly $3(d-1)$ facets, then G is a 3-regular bipartite graph on $d-1$ vertices. Conversely, it is, however, not true that any graph on $d-1$ nodes with constant degree 3 defines a face with $3(d-1)$ facets. See Figure 4.3 for an example.

5. Face Graphs with Many Nodes

Let $\mathcal{L}$ be the combinatorial type of a face of a Birkhoff polytope. The *Birkhoff dimension* $\mathrm{bd}(\mathcal{L})$ of $\mathcal{L}$ is the smallest n such that $\mathcal{L}$ is the combinatorial type of some face of $\mathcal{B}_n$. It follows from Theorem 2.6 that $\mathrm{bd}(\mathcal{L}) \leq 2d$ for a combinatorial type $\mathcal{L}$ of a d-dimensional face. In this section, we will study some properties of combinatorial types $\mathcal{L}$ of d-dimensional faces with $\mathrm{bd}(\mathcal{L}) \geq d$. In particular we will completely characterize those with $\mathrm{bd}(\mathcal{L}) \geq 2d-3$.

5.1. **Wedges.** In this section we will show that most faces of $\mathcal{B}_n$ are wedges over lower dimensional faces. The following main theorem characterizes graphs that correspond to wedges.

Theorem 5.1. *Let G be a face graph with $n \geq 3$ nodes in the upper layer and two connected adjacent nodes u and v of degree 2. Let G' be the graph obtained by attaching a path of length 3 to u and v. Then G is a face graph and the associated face is a wedge over the face of G.*

Proof. G' is clearly a face graph. We prove the theorem by induction over the dimension. The claim is true if G is the unique reduced graph on four nodes and four edges.

In the following we assume that the claim is true for face graphs defining a $(d-1)$-dimensional face of $\mathcal{B}_n$.

Let F be the face of G and F' that of G'. Let u' and v' be the two nodes added in G' and $e_1 = (u,v)$, $e_2 = (u',v')$, $f_1 = (u,v')$, and $f_2 = (u',v)$. See also Figure 5.1. Let G_1 be the face graph of all perfect matchings in G that do not contain e_1 (*i.e.*, the union of the perfect matchings in $M_{\neg e_1}$), and R the associated face of F (see Figure 5.2(a) and Figure 5.2(b)). As G has at least three nodes in each layer, this is a nonempty face. We claim that $F' = \mathsf{wedge}_R(F)$.

To show that F' is a wedge over F we have to show that F' has two facets F_1, F_2 isomorphic to F that meet in a face isomorphic to R, such that any other facet of F' is either

(1) a prism over a facet of F, or
(2) a wedge of a facet of F at a face of R,

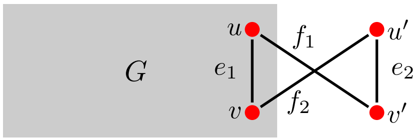


Figure 5.1. A wedge over a face graph G.

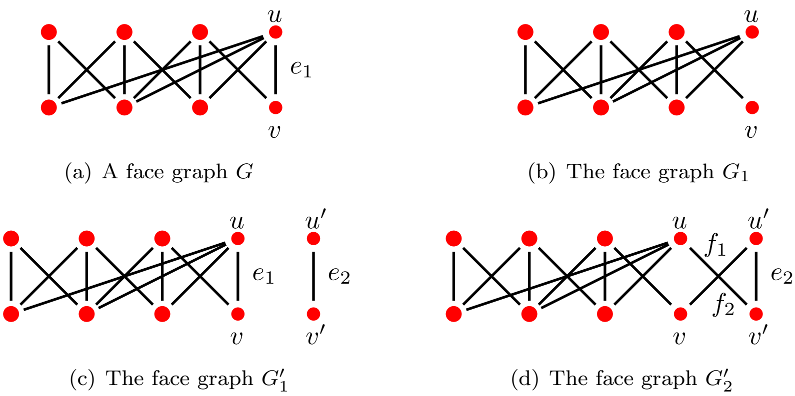
and, conversely, any facet of F (except possibly R) is used in one of these two cases.
Let G_1' be the graph obtained from G by adding u', v' together with e_2, and G_2' the graph obtained by removing e_1 and adding u', v' together with e_2, f_1 and f_2. See Figure 5.2(c) and Figure 5.2(d). Both graphs are subgraphs of G' and clearly define facets combinatorially isomorphic to F that intersect in R.
Let J' be a subgraph of G' that defines a facet of F'. There are three possibilities for edges contained in G' but not in J':

(1) both f_1 and f_2 are missing, or
(2) e_1 is missing, or
(3) e_1, e_2, f_1, and f_2 are present and some other edges are missing.

The first two cases are the two copies of F. In the last case, if u, u', v and v' form a connected component of J, then we have a prism over a facet of F sharing no vertex with R, and a wedge over a face of R by induction otherwise.
Conversely, let K be the graph of a facet S of F. If e_1 is missing in K, then R is a facet and corresponds to K. So we assume that e_1 is present in K. If one of u, v is connected to some other node in K (and, thus, necessarily also the other), then the wedge of K over the intersection of S and R is contained in G'. If u, v form a connected component, then S and R are disjoint and the prism over S is contained in F'. □

This gives a first classification of combinatorial types with Birkhoff dimension at least d.

Corollary 5.2. *Let $\mathcal{L}$ be the combinatorial type of a d-dimensional face of some $\mathcal{B}_n$. If* $\mathrm{bd}(\mathcal{L}) \geq d$*, then $\mathcal{L}$ is a wedge or a product.*



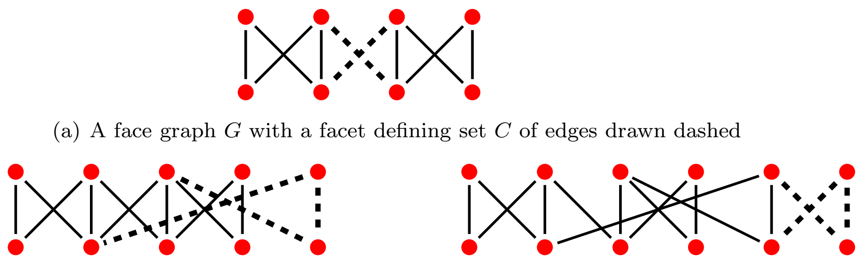
(a) A face graph G

(b) The face graph G_1

(c) The face graph G_1'

(d) The face graph G_2'

Figure 5.2. The various face graphs involved in the wedge construction



(a) A face graph G with a facet defining set C of edges drawn dashed

(b) The wedge over the facet defined by C

(c) The wedge over the complement of the facet defined by C

Figure 5.3. Wedge over a facet and its complement

Proof. Let F be a face of some $\mathcal{B}_n$ with combinatorial type $\mathcal{L}$, and assume that F is not a product. Let G be a reduced graph representing F. Thus, G is connected. Then $d = m - 2n + 1$ implies that each layer of G has at least one node of degree 2. By first completely reducing the graph G and subsequently resolving any multiple edges (similar to the proof of Theorem 2.6), we can assume that there are two adjacent nodes of degree 2. Now we can use the previous theorem. □

In fact, every d-dimensional combinatorial type $\mathcal{L}$ with $\mathrm{bd}(\mathcal{L}) \geq n$ is a wedge if its graph has a component with at least three nodes in each layer, *i.e.*, if the face is not a cube.

Theorem 5.3. *Let F be a face of $\mathcal{B}_n$. Then any wedge of F over a facet or the complement of a facet is also a face of some $\mathcal{B}_m$, $m \geq n$.*

Proof. Let G be an irreducible face graph corresponding to F. Let E be a facet of F with facet defining set C. Let e be any edge in C. The wedge over F is obtained by adding a path of length 3 to the endpoints of e, and the wedge over the complement is obtained by first replacing e by a path of length 3 and then adding another path of length 3 to the two new nodes. See Figure 5.3 for an illustration of the two operations. □

5.2. **Pyramids.** We want to discuss the structure of face graphs that correspond to pyramids. This will be important for the classification of faces with large Birkhoff dimension. In particular, we will see that for many faces F of $\mathcal{B}_n$ the pyramid over F is again face of a Birkhoff polytope.

Lemma 5.4. *Let G be a connected face graph containing an edge e that appears only in a single perfect matching M. Then M defines an edge of $\mathsf{F}(G)$ with any other perfect matching in G.*

Proof. Suppose not. Then there is a perfect matching M' such that $M \cup M'$ contains more than one cycle $C_1, \ldots, C_k$. The edge e is contained in such a cycle, as otherwise M' would use e. Assume this is C_1. However, using the edges of M' in $C_2, \ldots, C_k$ and the edges of M in C_1 defines another perfect matching M'' using e and different from both M and M'. This is a contradiction to the uniqueness of e. □

Theorem 5.5. *Let G be a connected irreducible face graph. Then $\mathsf{F}(G)$ is a pyramid if and only if G has an edge e that is contained in a unique perfect matching M.*

See Figure 5.4 for an example.

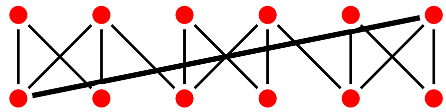


Figure 5.4. The thick long edge is contained in only one perfect matching in the graph. The graph defines a pyramid over a 3-cube.

Proof. If G has such an edge, then the union of all perfect matchings in G except M defines a proper face R of $\mathsf{F}(G)$ containing all but one vertex. Hence, $\mathsf{F}(G)$ must be a pyramid over S with apex M.
If $\mathsf{F}(G)$ is a pyramid, then let M be the perfect matching corresponding to the apex. Assume by contradiction that any edge e of M is contained in some other perfect matching M_e different from M. Let H be the subgraph defined by the union of these perfect matchings. Then $\mathsf{F}(H)$ is the smallest face S of $\mathsf{F}(G)$ containing those vertices. But H contains M, so S contains the apex. This is a contradiction, as then S is a pyramid with apex M whose base already contains all vertices corresponding to the M_e. □

Let G be a face graph with connected components $G_0, \dots, G_{k-1}$. We define the *circular connection* $\mathcal{C}(G)$ of G in the following way. For each $1 \leq i \leq k$, let u^i be a node in the upper and v^i a node in the lower layer of G_i. Then the nodes of $\mathcal{C}(G)$ are the disjoint union of the nodes of G_i, and the edges of $\mathcal{C}(G)$ are those of G_i together with edges from u^{i+1} to v^i for $0 \leq i \leq k-1$ (with indices taken modulo k). See Figure 5.5 for an illustration. Note that the circular connection is in general not a face graph. It is, if $G_i - \{u^i, v^i\}$ has a perfect matching for all i. If the perfect matchings in $G_i - \{u^i, v^i\}$ are also unique, then the circular connection is a face graph whose associated face is the pyramid over the face of G. This motivates the following definition.

Definition 5.6. Let G be a face graph with connected components $G_0, \dots, G_{k-1}$. A choice $S(G) := \{u^0, v^0, \dots, u^{k-1}, v^{k-1}\}$ of nodes $u^i, v^i \in G_i$ with u^i in the upper and v^i in the lower layer for $0 \leq i \leq k-1$ is *pyramidal* if the graph $G - S(G)$ has a unique perfect matching.

Corollary 5.7. *Let G be a connected irreducible face graph. If G has a node $u \in U$ and $v \in V$ such that $(u, v) \notin E$ and $G - \{u, v\}$ has a unique perfect matching, then $H := G + \{(u, v)\}$ defines a face graph that corresponds to the pyramid over $\mathsf{F}(G)$.*

Proof. $\{(u, v)\}$ is a facet defining set and the facet contains all but one vertex of H. □

Corollary 5.8. *Let G be a face graph with an edge e contained in a unique perfect matching. Then the pyramid over the face of G is again a face of G.*

Proof. The apex of a pyramid is the complement of a facet, and by Theorem 5.3 the wedge over any complement of a facet exists. □

5.3. **d-dimensional combinatorial types with Birkhoff dimension $\mathrm{bd}(\mathcal{L}) \geq 2d-2$.** For the remainder of this section we will study combinatorial types of faces with large Birkhoff dimension. We have seen above that for a given combinatorial type $\mathcal{L}$ this is bounded by $\mathrm{bd}(\mathcal{L}) \leq 2d$. The next proposition characterizes the case of equality.

Proposition 5.9. *Let $\mathcal{L}$ be a d-dimensional combinatorial type of a face of $\mathcal{B}_n$ with $\mathrm{bd}(\mathcal{L}) = 2d$. Then $\mathcal{L}$ is a cube.*

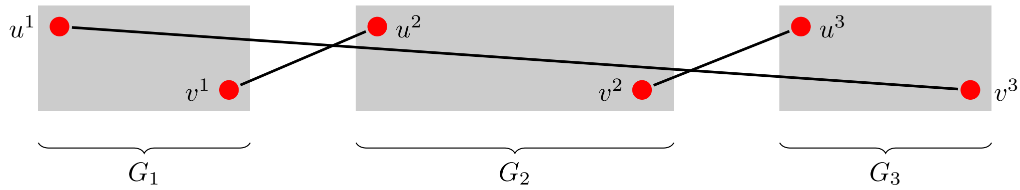


Figure 5.5. The circular connection of a graph.

Proof. We prove this by induction. If $d = 1$, then this follows from Proposition 3.7.
By Corollary 3.4 a connected irreducible graph of a d-dimensional face has at most $2d - 2$ nodes in each layer. Hence, the graph G of F must be disconnected. Let m, n, and k be its number of edges, nodes, and connected components, resp. Let G_1 be a connected component of G, and G_2 the remaining graph. Both are irreducible graphs. Let G_i have n_i nodes, m_i edges, k_i components and define a face of dimension d_i.
The dimension formula gives $d = m - 2n + k = m - 4d + k$, so $m = 5d - k$. We argue that $n_1 = 2$. Suppose otherwise. If $n_1 \geq 4$, then by Corollary 3.6 we can estimate

$$5d - k \;=\; m \;\geq\; 2n_1 + \left\lceil \frac{n_1}{2} \right\rceil + 2n_2 + \left\lceil \frac{n_2}{2} \right\rceil \;\geq\; 2n + \frac{n}{2} = 5d\,.$$

As $k \geq 1$ this is not possible. Now if $n_1 = 3$, then $n_2 = 2d - 3$, and by Proposition 3.7 we know $m_1 \geq 7$. So we can compute

$$5d - k \;=\; m \;\geq\; 7 + 2n_2 + \left\lceil \frac{n_2}{2} \right\rceil \geq\; 7 + 4d - 6 + \frac{2d-2}{2} = 5d\,.$$

which again contradicts $k \geq 1$. So $n_1 = 2$, and G_1 defines a segment. G_2 now is an irreducible graph of dimension $d - 1$ on $2(d - 1)$ nodes. By induction, this must be a cube. □

Proposition 5.10. *A d-dimensional combinatorial type $\mathcal{L}$ of a face of a Birkhoff polytope with $\operatorname{bd}(\mathcal{L}) \geq 2d - 1$ is a product of a cube and a triangle.*

Proof. As in the previous proof, our graph has $d + 2n - k = d + 4d - 2 - k = 5d - 2 - k$ edges. Let $n_1, \ldots, n_k$ be the number of nodes of the upper layer of each component of the graph. Let $k_{2/3}$ be the number of components with two or three nodes and k_o the number of components with an odd number of nodes. Using Corollary 3.6 we can estimate the number of edges in the graph by

$$e \geq \sum_{i=1}^{k} \left(2n_i + \left\lceil \frac{n_i}{2} \right\rceil \right) - k_{2/3} \;=\; 4d - 2 + d - \frac{1}{2} + \frac{k_o}{2} - k_{2/3} \;=\; 5d - \frac{5}{2} + \frac{k_o}{2} - k_{2/3}\,.$$

Hence,

$$e \;=\; 5d - k - 2 \;\geq\; 5d - \frac{5}{2} + \frac{k_o}{2} - k_{2/3} \qquad \Longleftrightarrow \qquad k \leq k_{2/3} - \frac{k_o - 1}{2}\,,$$

and we conclude $k_{2/3} = k$ and $k_o \leq 1$. So at most one component has more then two nodes on each layer. However, $2d - 1$ is odd, hence $k_o = 1$. This implies the proposition. □

Proposition 5.11. *A combinatorial type $\mathcal{L}$ of a Birkhoff face F of dimension $d \geq 3$ with $\operatorname{bd}(\mathcal{L}) \geq 2d - 2$ is either a product of two lower dimensional faces or a pyramid over a cube of dimension $d - 1$. For 1-dimensional and 2-dimensional types $\mathcal{L}$ we have $\operatorname{bd}(\mathcal{L}) = 3$.*

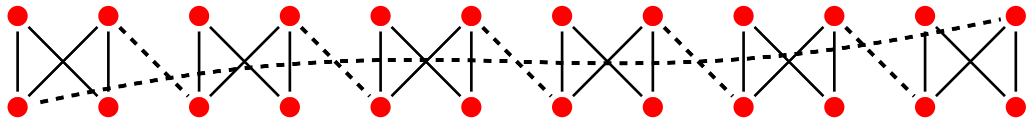


Figure 5.6. The only connected irreducible face graph on $2d-2$ nodes (here $d = 7$). The face set defining the base is drawn with dashed lines. Those edges appear in the unique matching corresponding to the apex.

Proof. Assume that F is not a product. We first consider the case that $d \geq 4$. In this case F corresponds to an irreducible face graph G on $2d-2$ nodes with $e = d+2(2d-2)-1 = 5d-5$ edges. By Proposition 3.5 the maximum degree of a node in the graph is 3. Hence, in each layer we have $d-1$ nodes of degree 2 (*i.e.*, minimal nodes, as G is irreducible) and $d-1$ nodes of degree 3.

Consider the nodes in the upper layer. By Proposition 3.2 a node of degree 3 is partner for precisely one minimal node, and any minimal node has a unique partner. As also in the lower layer the degree of a node is at most 3 we can deduce that a node in the lower layer is on at most one path connecting a minimal node with its partner. By counting nodes we see that each node in the lower layer is on precisely one such path.

We split the graph according to these paths. Let $u_1, \dots, u_{d-1}$ be the minimal nodes in the upper layer, and for each $1 \leq i \leq d-1$ define N_i to be the graph induced by u_i, its unique partner, and the two paths of length 2 between them. By the above argument, the graphs N_i are pairwise disjoint. Hence, their union $N := \bigcup N_i$ define a face subgraph of G whose corresponding face is the product of $d-1$ segments, *i.e.*, it is isomorphic to a $(d-1)$-dimensional cube.

The graphs N and G have the same number of nodes, and G has $d-1$ additional edges. N is disconnected, so those $d-1$ edges must connect the $d-1$ components of N. As G is a face graph, *i.e.*, each edge must be contained in some perfect matching of G, the graphs N_i can only be connected circularly. Hence, up to relabeling and flipping upper and lower layer in the graphs N_i, the graph G must look like Figure 5.6. This is the circular connection of the N_i.

For $d = 3$ and $n = 4$ there is a second irreducible graph on four nodes, see Figure 5.7(a). This graph defines a tetrahedron. However, the graph in Figure 5.7(b) also defines a tetrahedron, so this face already appears in $\mathcal{B}_3$. □

We can combine the three previous Propositions 5.9, 5.10, and 5.11 into the following slightly extended theorem.

Theorem 5.12. *Let $\mathcal{L}$ be a combinatorial type of a d-dimensional face of a Birkhoff polytope with* $\mathrm{bd}(\mathcal{L}) \geq 2d-2$. *Then $\mathcal{L}$ is a*

(1) a cube, if $\mathrm{bd}(\mathcal{L}) = 2d$,

(2) a product of a cube and a triangle, if $\mathrm{bd}(\mathcal{L}) = 2d-1$,

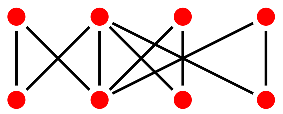
(3) a polytope of one of the following types, if $\mathrm{bd}(\mathcal{L}) = 2d-2$:

  (a) a pyramid over a cube,

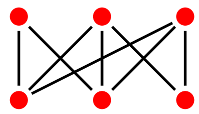
  (b) a product of a cube and a pyramid over a cube,

  (c) a product of two triangles and a cube.

Proof. The only claim not contained in the previous propositions is the classification of products leading to a d-dimensional face F on $2d-2$ nodes. Assume that F is a product



(a) A second 3-dimensional face with an irreducible graph on four nodes.



(b) A smaller representation of the same face.

Figure 5.7. Two representations of the same combinatorial type of face.

$F_1 \times F_2$ of polytopes F_1, F_2 (which may itself be products) of dimensions d_1 and d_2. Let $\Gamma(F_i)$ have n_i nodes in each layer. Then $d_1 + d_2 = d$. Define non-negative numbers $r_i := 2d_i - n_i$, $i = 1, 2$.
Assume that $\Gamma(F)$ has k and $\Gamma(F_i)$ has k_i components, $i = 1, 2$. Then $k_1 + k_2 = k$, $\Gamma(F)$ has $2n + d - k = 2(2d-2) + d - k = 5d - 4 - k$ edges, and $\Gamma(F_i)$ has

$$2n_i + d_i - k_i \;=\; 2(2d_i - r_i) + d_i - k_i \;=\; 5d_i - 2r_i - k_i$$

edges, for $i = 1, 2$. This implies $r_1 + r_2 = 2$. Hence, $r_1 = 2, r_2 = 0$ or $r_1 = r_2 = 1$ or $r_1 = 0, r_2 = 2$ and the claim follows by induction. □

5.4. **d-dimensional combinatorial types with Birkhoff dimension at least $2d-3$.** In this section we introduce a new construction for polytopes, the *joined products* and *reduced joined products* and use them to classify faces of $\mathcal{B}_n$ for $n = 2d-3$, but also many other faces of $\mathcal{B}_n$ are of this type. We give a combinatorial description and deduce their corresponding face graph. We use these to classify combinatorial types of faces in Theorem 5.16.
Let $Q_1, \dots, Q_k$ be polytopes in $\mathbb{R}^m$ (not necessarily all m-dimensional). The *Cayley sum* of $Q_1, \dots, Q_k$ is the polytope

$$\mathrm{Cay}(Q_1, \dots, Q_k) \;:=\; \mathsf{conv}(Q_1 \times e_1,\, Q_2 \times e_2,\, \dots, Q_k \times e_k)\,,$$

where $e_1, e_2, \dots, e_k$ are the k-dimensional unit vectors. We use this construction for a special set of polytopes $Q_1, \dots, Q_k$. Let $\mathbf{0}^{(d)}$ be the d-dimensional zero vector, and P_i d_i-dimensional polytopes for $1 \le i \le k$. We define

$$Q_i \;:=\; P_1 \times \cdots \times P_{i-1} \times \mathbf{0}^{(d_i)} \times P_{i+1} \times \cdots \times P_k\,,$$

and

$$Q_0 \;:=\; P_1 \times \cdots \times P_k\,.$$

Definition 5.13. The *joined product* of the polytopes $P_1, \dots, P_k$ is

$$\mathcal{JP}(P_1, \dots, P_k) \;:=\; \mathrm{Cay}(Q_1, \dots, Q_k)\,, \tag{5.1}$$

and the *reduced joined product* is

$$\mathcal{JP}^{\mathrm{red}}(P_1, \dots, P_k) \;:=\; \mathrm{Cay}(Q_0, \dots, Q_k)\,. \tag{5.2}$$

The reduced joined product is the special case of the joined product where one of the factors is just a point. Hence, in the following considerations on combinatorial properties we restrict to joined products.
Let m_i be the number of vertices of P_i, and $M := \prod_{i=1}^k m_i$. We will show that the joined product P of $P_1, \dots, P_k$ has $\sum_{i=1}^k \frac{M}{m_i}$ vertices. Assume by contradiction that there is $v \in \bigcup_{i=1}^k \mathbf{V}(Q_i) \times e_i$ that is not a vertex of P. Then v is a convex combination of some of the

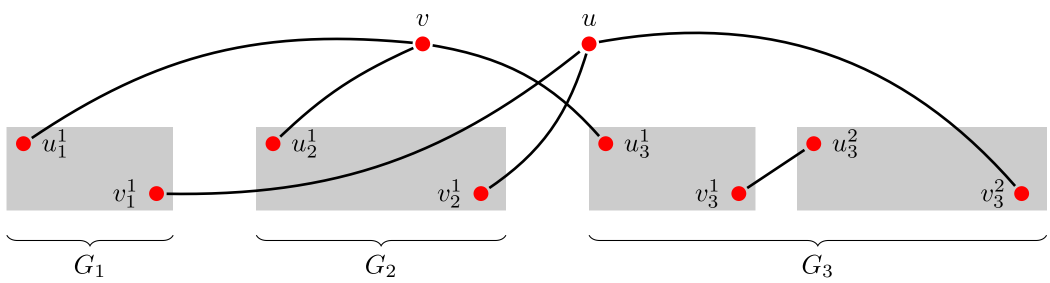


Figure 5.8. The joined product of three graphs. The reduced joined product additionally has an edge between u and v.

other vertices, say $v_1, \ldots, v_r$. The point v, and any v_j, $1 \leq j \leq r$ has exactly one entry different from 0 among the last k coordinates. Hence, any points in the convex combination of v with a positive coefficient coincide with v in that entry. By construction, this implies that v and all points in its convex combination are completely contained in one the factors $Q_i \times e_i$, for some i. But v and v_j, $1 \leq j \leq r$ are vertices of $Q_i \times e_i$, a contradiction. Hence, any vertex of some Q_i corresponds to a vertex of the joined product.

The joined products have two obvious types of facets. For any facet F_1 of P_1 the polytope $\mathcal{JP}(F_1, P_2, \ldots, P_k)$ is a facet of P. We can do this for any factor of the product and any two such facets are distinct. For $k \geq 3$ we also have the facets

$$\mathsf{conv}(Q_1 \times e_1, \ldots, Q_{i-1} \times e_{i-1}, Q_{i+1} \times e_{i+1}, \ldots, Q_k \times e_k)\,.$$

We construct the corresponding graphs. Let $G_1, \ldots, G_k$, $k \geq 2$ be (not necessarily connected) face graphs. Let H be a graph with two isolated nodes, and $\overline{H}$ a graph with two nodes and the edge between them. We define the *joined product* $\mathcal{JPG}(G_1, \ldots, G_k)$ of $G_1, \ldots, G_k$ as the union of the circular connections of H with each G_i, and the *reduced joined product* $\mathcal{JPG}^{\text{red}}(G_1, \ldots, G_k)$ as the union of the circular connections of $\overline{H}$ with each G_i.

We make this construction more precise. For each graph G_i with connected components $G_i^1, G_i^2, \ldots, G_i^{c_i}$ we select a set $S(G_i)$ of a pair of nodes u_i^j, v_i^j in each G_i^j, $1 \leq j \leq c_i$, with u_i^j in the upper and v_i^j in the lower layer. Let u, v be the nodes of H. Then the *joined product* $\mathcal{JPG}(G_1, \ldots, G_k)$ is the disjoint union of H and $G_1, \ldots, G_k$ together with the edges $(u, v_i^{c_i})$ and (v, u_i^1) for $1 \leq i \leq k$ and (v_i^j, u_i^{j+1}) for $1 \leq i \leq k$ and $1 \leq j \leq c_i - 1$. The *reduced joined product* is obtained in the same way with $\overline{H}$ instead of H (with labels u and v for the nodes of $\overline{H}$). See Figure 5.8 for an illustration.

Clearly, the isomorphism types of the resulting graphs depend on the choice of the two nodes in each component of the G_i. In general, they will not be face graphs. More precisely, the joined product graph $\mathcal{JPG}(G_1, \ldots, G_k)$ is a face graph if and only if for each $1 \leq i \leq k$ the circular connection of H and G_i using the nodes in $S(G_i)$ is a face graph, and similarly for $\mathcal{JPG}^{\text{red}}(G_1, \ldots, G_k)$. Note that we have called a choice $S(G_i)$ of nodes u_i^j, v_i^j, $1 \leq j \leq c_i$ *pyramidal*, if $G_i - S(G_i)$ has a unique perfect matching, for $1 \leq i \leq k$, see Definition 5.6.

Theorem 5.14. *Let $G_1, \ldots, G_k$ be face graphs with pyramidal sets $S_i(G_i)$ of nodes and $F_i := \mathsf{F}(G_i)$, $1 \leq i \leq k$.*

(1) $G := \mathcal{JPG}(G_1, \ldots, G_k)$ is a face graph with face given by $\mathcal{JP}(F_1, \ldots, F_k)$.

(2) $G := \mathcal{JPG}^{\text{red}}(G_1, \ldots, G_k)$ is a face graph with face given by $\mathcal{JP}^{\text{red}}(F_1, \ldots, F_k)$.

Proof. We prove only the first statement. The proof of the second is analogous.

Let H be the graph with two isolated nodes u and v as above. By construction, for each i the circular connection of the disjoint union of H and G_i has a unique perfect matching M_i. This matching is given by the edges $(u, v_i^{c_i})$, (v, u_i^1), the edges (v_i^j, u_i^{j+1}) for $1 \leq j \leq c_i - 1$ and the unique perfect matching in $G_i - S(G_i)$. Hence, G is a face graph. Its perfect matchings are precisely products of some M_i with a choice of a perfect matching in all G_j for $j \neq i$.
It remains to show that convex hull of the vertices defined by the perfect matchings in the joined product of the graphs is affinely isomorphic to the joined product of the F_i. For this, it suffices to note that a perfect matching that contains, for some $1 \leq i \leq k$, one of the edges $(u, v_i^{c_i})$, $(v, u_i^1$ or (v_i^j, u_i^{j+1}) for $1 \leq j \leq c_i - 1$ necessarily also contains the others. Hence, up to affine isomorphism, we can forget all but one of the corresponding coordinates. This gives the Cayley structure (5.1) with the products of the remaining F_j, $j \neq i$. ☐

Note that, as a face can have more than one representation as an irreducible face graph, it does not follow from this theorem that all graphs of faces of some $\mathcal{B}_n$ that are joined products of some other faces are of the form given in the theorem. However, if a face is a joined product of some polytopes, then those are again faces of some Birkhoff polytope. We need one more lemma before we can continue our classification.

Lemma 5.15. *Let G be an irreducible face graph with a minimal node v in the upper layer. Let w_1, w_2 be the neighbors of v. If $x \neq v$ is a node adjacent to w_1 but not to w_2, then the graph G' obtained by replacing (x, w_1) with (x, w_2) is a face graph with the same combinatorial type as G.*

Proof. The node v has degree 2 in both graphs, and the reduction at v gives the same graph for G and G'. ☐

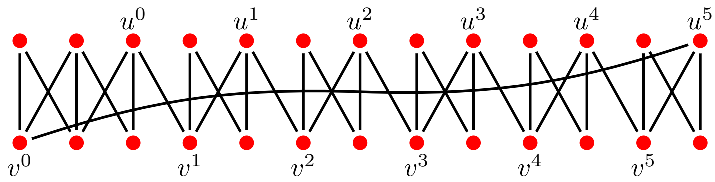
We are ready to classify all d-dimensional combinatorial types $\mathcal{L}$ with $\mathrm{bd}(\mathcal{L}) \geq 2d - 3$.

Theorem 5.16. *Let $\mathcal{L}$ be a combinatorial type of a d-dimensional face with $\mathrm{bd}(\mathcal{L}) \geq 2d-3$. Then $\mathcal{L}$ is of one of the following four types.*

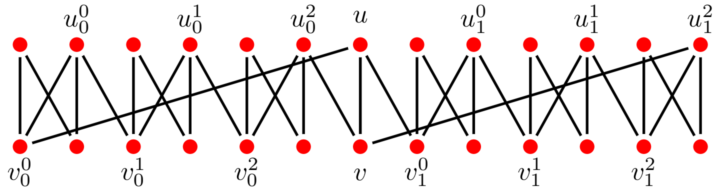
(1) Pyramid over a product of a cube and a triangle. See Figure 5.9(a).
(2) A reduced joined product of two cubes (of possibly different dimensions). See Figure 5.9(b).
(3) A joined product of three cubes (of possibly different dimensions). See Figure 5.9(c).

Note that the theorem does not claim that these faces do really only appear in $\mathcal{B}_{2d-3}$, but only that, if a face appears in $\mathcal{B}_{2d-3}$ for the first time, then it must be of one of these types. The stronger statement is certainly true for the first cases, as it contains the product of a $(d-3)$-cube with a triangle as a proper face, and this cannot be represented with less nodes.

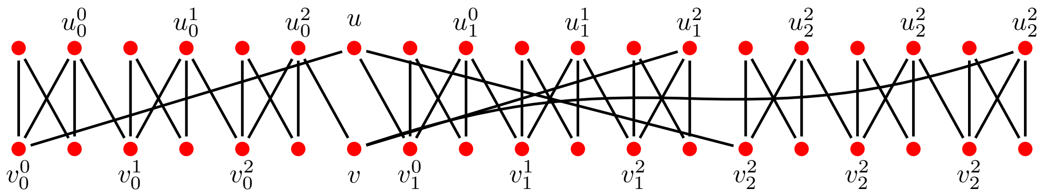
Proof. We classify the possible face graphs. The previous theorem translates this into combinatorial types of faces. An irreducible face graph on $2d - 3$ nodes has $5d - 7$ edges. We have either $d-1$ or $d-2$ minimal nodes, and the maximal degree of a node is 4. By counting we conclude that there is at most one node of degree 4 in each layer. Further, if there is a node of degree 4 in one layer, then each node of degree at least 3 in this layer is partner for some minimal node. In the other case at most one of the nodes of degree 3 is not a partner. We show first that we can reduce to the case that the maximal degree in G is 3.
Let v be a node of degree 4 in the upper layer with neighbors $w_1, \ldots, w_4$. Then v is partner for two nodes u_1, u_2. For both there are two disjoint paths of length 2 connecting them to v. Let w_1, w_2 be the intermediate nodes of the paths to u_1. The intermediate nodes of the paths to u_2 cannot coincide with those two, as otherwise (v, w_1) and (v, w_2) are not part of a perfect matching. So we are left with the cases sketched in Figures 5.10(a) and 5.10(b), up to additional edges incident to some of the w_i.



(a) d-faces on $2d-3$ nodes: type (a),



(b) d-faces on $2d-3$ nodes: type (b)

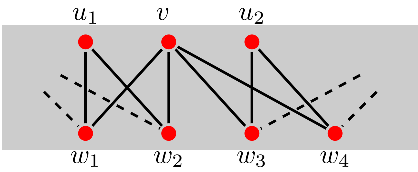


(c) d-faces on $2d-3$ nodes: type (c)

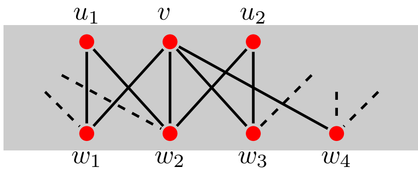
Figure 5.9. d-dimensional face graphs on $2d-3$ nodes.

Consider first the case given in Figure 5.10(a). Assume that the degree of $w_1, \dots, w_4$ is at least 3. As all but at most one node are partner for some minimal node, and at most one node has degree 4, we can pick one of $w_1, \dots, w_4$ that has degree 3 and is a partner for some minimal node u. Assume this is w_1. We need two different paths of length 2 connecting w_1 to its partner. Hence, one of the paths must use one of the edges (w_1, u_1) or (w_1, v). Thus, the partner u must be one of the nodes w_2, w_3, w_4. This contradicts the assumption that all four nodes have degree at least 3. So at least one of w_1, w_2, w_3, w_4 has degree 2. Assume that this is w_2. Hence, we can use Lemma 5.15 for w_2 and u_1, v to move one of the edges incident to v to u_1 to obtain a graph with maximal degree 3 in the upper layer. See Figure 5.11. By our assumption that the corresponding combinatorial type of the face has Birkhoff dimension $\operatorname{bd}(\mathcal{L}) \geq 2d-3$ we know that the graph remains irreducible.

Now consider the graph given in Figure 5.10(b). By the same argument as above at least one of $w_1, \dots, w_4$ has degree 2. If this is w_1 or w_3 we can proceed as above and move an edge incident to v to either u_1 or u_2. So assume that only w_4 has degree 2. So w_1, w_2, w_3 have degree at least 3. If w_2 has degree 4, then it is partner for two minimal nodes. So at least one of w_1, w_3 would have degree 2, contrary to our assumption. So w_2 has degree 3. If it were a partner for some minimal node, then this would have to be w_1 or w_3, again contradicting our assumption. So w_2 is not a partner of some minimal node. By assumption this implies that the maximal degree in the lower layer is 3, and both w_1 and w_3 are partner



(a) node of degree 4, first case



(b) node of degree 4, second case

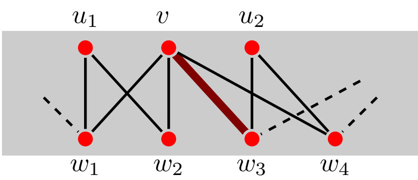
Figure 5.10. The two cases of a node of degree 4 in Theorem 5.16. The dashed partial edges indicate that there may or may not be additional edges incident to some of the w_i.

of some minimal node. By construction, this can only be w_4. But, again by assumption, a minimal node has a unique partner, so this case does not occur.

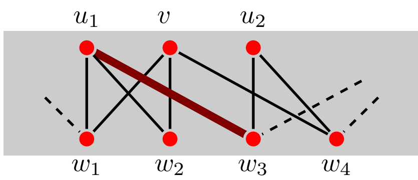
We can repeat the same argument for the lower layer. This transforms G into a face graph whose combinatorial type is combinatorially isomorphic to the original one, but the graph has maximal degree 3 in both layers.

Hence, in the following we can assume that the maximal degree of a node in G is 3. In that case the graph necessarily has $d-2$ minimal nodes in each layer, and $d-1$ nodes of degree 3. Pick a partner p_i^u for each minimal node x_i^u in the upper, and p_i^l for each minimal node x_i^l in the lower layer, $1 \leq i \leq d-2$. The p_i^u are pairwise distinct as a node of degree $k \geq 3$ is partner for at most $k-2$ nodes (unless it is the only partner in the graph, see Proposition 3.2). See also Figure 5.12 for two examples. Let y_u and y_l be the remaining node in each layer. Let N_i be the induced graph on p_i^u, x_i^u and the two paths of length 2 between them. The N_i, $1 \leq i \leq d-2$, are pairwise disjoint, as otherwise there would be a node of degree 4 in the lower layer. Let z_l be the node in the lower layer not contained in any N_i. We distinguish various cases.

(1) y_u and z_l are connected, and z_l is minimal. See Figure 12(a). In this case z_l has a partner p_l contained in some N_i. We may assume that this is N_1. We replace N_1 by the graph induced by N_1, y_u and z_l. Then N_1 has 6 nodes and at least 7 edges. Hence, $N := \bigcup N_i$ defines a face subgraph of G with the same number of nodes, with $d-2$ components, and at most $d-2$ edges less than G. Thus, N and G differ by precisely $d-2$ edges. As any edge must be contained in a perfect matching, those edges must connect the components of H in a circular way. Further, y_u and y_l have degree 2 in

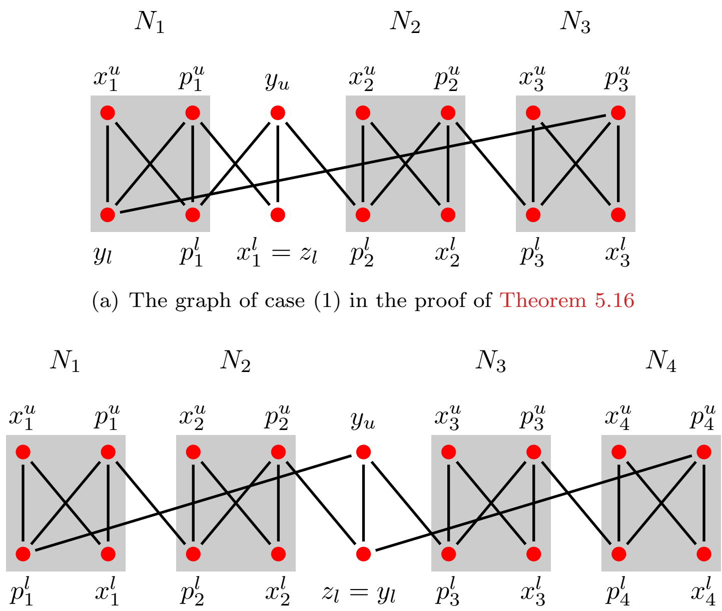


(a) w_2 has degree 2.



(b) Rearranged edges at v.

Figure 5.11. Moving one edge incident to v in the proof of Theorem 5.16. The dashed partial edges indicate that there may or may not be additional edges incident to some of the w_i.



(a) The graph of case (1) in the proof of Theorem 5.16

(b) The graph of case (2) in the proof of Theorem 5.16

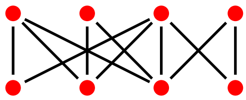
Figure 5.12. Decomposing graphs in the proof of Theorem 5.16

N_1, but degree 3 in G, so $N_1 - \{y_u, y_l\}$ has a unique perfect matching. Hence, G is a pyramid over N, and N is a product of segments and a triangle.

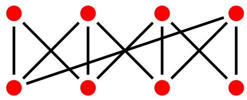
(2) y_u and z_l are connected, and both have degree 3. See Figure 12(b). In that case, $z_l = y_l$. Let N_0 be the subgraph induced by y_u and y_l (with one edge). Then $N := \bigcup N_i$ is a face subgraph of G with $4d - 7$ edges. The only way to obtain a connected irreducible face graph from N by adding d edges is to split the set of N_i, $i \geq 1$ into two nontrivial sets and connect both with N_0 circularly. This gives a reduced joined circular product of two cubes (not necessarily of equal dimension)

(3) y_u and z_l are not connected. Assume that the degree of z_l is 2. Then $z_l \neq y_l$, and it needs a partner in the lower layer. The two edges incident to z_l cannot both end in the same N_i, as one node in the upper layer of each N_i has degree 2. Hence, the two incident edges end in different N_1, say at nodes $s_1 \in N_1$ and $s_2 \in N_2$. Yet, z_l needs a partner, so there is either an edge from s_1 to a node of N_2 or from s_2 or a node of N_1. Hence, either s_1 or s_2 have degree 4. By construction, such a node does not exist, so we can assume that z_l has degree 3. Again, the edges incident to z_l necessarily end in different N_i, as one node in the upper layer of each N_i has degree 2. Hence, z_l cannot be a partner, so $z_l = y_l$.

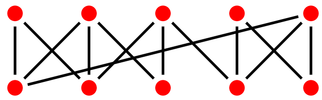
The graph $N := \bigcup N_i \cup \{y_u, y_l\}$ has $4d - 8$ edges. G has $d + 1$ additional edges, and as G is a connected face graph each N_1 is incident to at least two of them (as, in particular, each edge must be contained in a perfect matching). But as y_u, y_l have degree 3 we conclude that each N_i is incident to exactly two of the $d + 1$ additional edges. Hence, as before, the only way to create a connected face graph by adding only $d + 1$ edges is to split the set of the N_i into three nontrivial parts and connect them to y_u, y_l circularly. □



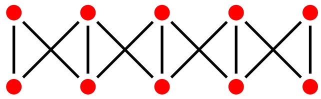
(a) simplex



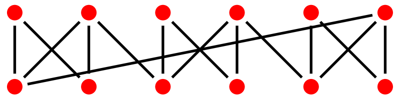
(b) edge ⋆ square



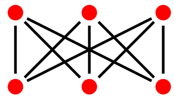
(c) pyramid over triangle prism



(d) wedge of a square pyramid over an edge



(e) pyramid over cube



(f) Birkhoff polytope

Figure 7.1. The 4-dimensional Birkhoff faces which are not products

6. Low-dimensional Classification

For a classification of faces of a given dimension d it is essentially sufficient to classify those faces that have a connected face graph. The others are products of lower dimensional faces of the Birkhoff polytope, and they can thus be obtained as pairs of face graphs of a lower dimension.

The three dimensional faces have been classified before by several others, see, *e.g.*, [12] or [7]. By Theorem 2.6 we know that any d-dimensional face appearing in some Birkhoff polytope does so in a Birkhoff polytope $\mathcal{B}_n$ for $n \leq 2d$.

We have implemented an algorithm that generates all *irreducible connected* face graphs of a given dimension and with a given number of nodes. The implementation is done as an extension [36] to `polymake` [28]. The algorithm provides a method `generate_face_graphs` that takes two arguments, the number of nodes of the graph in one of the layers, and the dimension of the face. It constructs all irreducible face graphs with this number of nodes and the given dimension, up to combinatorial isomorphism of the corresponding face (as some combinatorial types have irreducible face graphs with different number of nodes they can appear in several times different runs of the method). Dimension and number of nodes fixes the number of edges, and, roughly, the method recursively adds edges to an empty graph until it reaches the appropriate number.

It distinguishes between graphs of minimal degree 3 and those with at least one node of degree 2. Constructing those with minimal degree 3 is simple, as filling each node with at least three edges does not leave much choice for a face graph. This can be done by a simple recursion using some of the results in Section 3 and Section 4.

For the other graphs we iterate over the number of nodes of degree 2, and first equip each such node with a partner and the necessary edges, and add further edges until all remaining nodes have degree 3. Here we use the results of Section 3 and Section 4 to prune the search tree at an early stage if graphs in this branch either will not be irreducible or not a face graph. The few remaining edges are then again filled in recursively. See also the comments in the code.

The data in `polymake` format can be obtained from [35]. The following theorem summarizes the results. For the product types we have just counted the non-isomorphic products of connected irreducible graphs.

dim	1	2	3	4	5	6	7	8
# non-product types	1	1	2	6	20	86	498	3712
# product types	0	1	3	5	13	43	163	818
# pyramids	1	2	2	4	10	28	98	416

Table 6.1. Low dimensional faces of Birkhoff polytopes. The last row of the table collects the number of pyramids among the non-product types.

Theorem 6.1. *(1) In dimension* 2 *there are two combinatorial types of faces, a square and a triangle.*

(2) In dimension 3 *there are two combinatorial types that are products of lower dimensional faces, and two other types, the* 3*-simplex and the pyramid over a square.*

(3) In dimension 4 *there are five combinatorial types that are products, and six other types: (*a*) a simplex, (*b*) the join of a segment and a square, (*c*) a wedge W_1 over an edge of the base of a square pyramid, (*d*) the Birkhoff polytope $\mathcal{B}_3$, and (*e*) the pyramids over a cube and a triangle prism. See Figure 7.1 for examples of face graphs for the non-product types.*

(4) In dimension 5 *there are 13 combinatorial types that are product, and 20 other types: (*a*) the pyramids over all* 4*-dimensional types except $\mathcal{B}_3$, (*b*) the join of two squares, the wedges over a facet and an edge of $\mathcal{B}_3$, (*c*) the wedge over the complement of the square pyramid in W_1, (*d*) the wedge over a* 3*-simplex in W_1, (*e*) the wedge over the complement of a* 3*-simplex in W_1 and its dual, (*f*) the wedge over the complement of a triangle prism in W_1, (*g*) the wedge over a triangle of the prism over a triangle in the pyramid over a triangle, (*h*) the wedge over the edge of a square in the double pyramid over the square, and (*i*) the join of two squares.*

(5) In dimension 6 *there are 43 combinatorial types that are product, and 86 other types.*

(6) In dimension 7 *there are 163 combinatorial types that are product, and 498 other types.*

(7) In dimension 8 *there are 818 combinatorial types that are product, and 3712 other types.*

The descriptions given in the theorem are not unique. Table 6.1 summarizes this theorem.

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