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FACETS IN THE CROSSING NUMBER POLYTOPE*

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Abstract. In the last years, several integer linear programming (ILP) formulations for the crossing number problem arose. While they all contain a common conceptual core, the properties of the corresponding polytopes have never been investigated. In this paper, we formally establish the crossing number polytope and show several facet-defining constraint classes.

Key words. crossing number, integer linear program, polyhedral study, facets, Kuratowski constraints

AMS subject classifications. 05C10, 65K05, 90C57

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1. Introduction. The crossing number $\mathrm{cr}(G)$ is the smallest number of pairwise edge crossings over all drawings of a graph G into the plane. The problem has been investigated for over 60 years (see [26] for an extensive bibliography), yet, still surprisingly little is known about this most traditional nonplanarity measure of graphs. Even the crossing number for arbitrarily sized complete and complete bipartite graphs—which constitute the historically first research questions in this area—can currently only be conjectured; see, e.g., [24]. The problem is nondeterministic polynomial time hard (NP-hard) [13], even for graphs with maximum degree 3 [16]. Its approximability in unknown, even for degree-bounded graphs; the known results all consider only restricted graph classes [5, 9, 14, 17, 18, 19], and sometimes even do not directly approximate $\mathrm{cr}(G)$ but $|V| + cr(G)$ within a nonfixed factor [1, 12]. Yet, it is known that the problem is fixed parameter tractable in linear time [15, 20], even though the corresponding algorithms are not practical.

In the last years, several exact integer linear programming (ILP) approaches, focusing on "real-world graphs" for graph drawing and diagramming applications, tackled the crossing number problem [2, 3, 8, 11] and some of its variants [4, 7, 10]. Based on a natural but practically useless formulation, two competing extended ILP formulations were introduced. Using branch-and-cut-and-price techniques (see, e.g., [27] for a general description of this concept), they allow the computation of the exact crossing number for sparse, medium sized graphs. Yet, they were only evaluated experimentally, and no research has yet been conducted regarding the problem's polyhedral properties or the formal strength of certain constraint classes. In fact, there is not even a general definition of the polytope over which to optimize.

In section 2, we introduce the notion of the *crossing number polytope* $\mathcal{P}_{\mathrm{cr}}$ using only natural variables, and describe the known formulations in the context thereof. Thereby, the paper aims to give an overview over the known ILP formulations from a different and more abstract perspective than was achievable by the publications dealing with the individual formulations. Section 3 then establishes several constraint

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classes that can be used in the context of $\mathcal{P}_{\mathrm{CR}}$, as well as in the extended formulations. In the sections thereafter, we show that these constraints define *facets*, which are the first such proofs in the context of crossing numbers. The strength of these constraints is particularly interesting, as in practice one often tends to be relatively dissatisfied with their performance: We require very many of them, and the observed benefit of adding such constraints (especially in later iterations of the algorithm) can be very limited. Therefore it has been suggested that they could be theoretically weak, and one should probably look for strengthened versions of them.

2. Crossing number polytope. The drawing of a graph is a mapping of nodes and edges into points and curves into the plane. Drawings of planar graphs can be categorized into equivalence classes, so-called *embeddings*, that abstract from the exact point sets to only specify the "general layout" by specifying the cyclic order of the edges around their respective nodes, thereby defining *faces*, i.e., regions bounded by edges. Given an abstract embedding Γ of a graph G, it is straight-forwardly possible to generate some (though not necessarily aesthetically pleasing) drawing of G adhering to Γ.

In the following, let $G = (V, E)$ be the given (nonplanar) graph. When considering a drawing $\mathcal{D}$ of a nonplanar graph G, we can define a *planarization* P of G as the planar graph obtained from G by replacing the crossings in $\mathcal{D}$ by dummy nodes of degree 4. Formally, we say $P = (V \cup D, E')$ is a planarization of G if (i) P is planar and (ii) we can obtain P from G by iteratively replacing pairs of edges (say $e = \{u_e, v_e\}, f = \{u_f, v_f\}$) by a star, i.e., four edges connecting each of u_e, v_e, u_f, v_f with a (unique) node in D.

Drawing a nonplanar graph can then be seen as (a) finding a planarization P of G, (b) finding an embedding Γ_P of P, (c) finding a drawing for Γ_P, and (d) drawing G consistent with this latter drawing. In the context of crossing minimization, we are exclusively interested in step (a), as the other steps are trivial when P is a feasible planarization (disregarding further aesthetics criteria): Given a nonplanar graph G, find a planarization P of G with as few dummy nodes (D) as possible.

Given some set S, we use $S^{\{k\}} := \{S' \subseteq S \mid |S'| = k\}$ and $S^{\langle k \rangle} := \{(a_1, \ldots, a_k) \mid \forall 1 \leq i < j \leq k : a_i, a_j \in S \wedge a_i \neq a_j\}$ to denote the sets of all unordered and ordered, respectively, k-tuples with pairwise disjoint elements of S. Since an edge is a two-element set of vertices, we can write

$$\mathcal{CP} := \Big\{\{e, f\} \in E^{\{2\}} \ : e \cap f = \emptyset\Big\}$$

to define the set of all edge pairs that may cross in any optimal drawing of G; we know that adjacent edges will never cross. The central underlying idea of all current crossing minimization ILPs is to introduce variables

$$x_{\{e,f\}} \in \{0, 1\} \quad \forall \{e, f\} \in \mathcal{CP} \tag{1}$$

that are 1 if the edges e and f cross, and 0 otherwise.[1] The objective function can then be written as $\min \sum_{\{e,f\} \in \mathcal{CP}} x_{\{e,f\}}$; i.e., we want to minimize the number of crossings. By introducing objective function coefficients $c_{\{e,f\}} = w(e) \cdot w(f)$ for each variable $x_{\{e,f\}}$, we can easily model the weighted crossing number problem with the edge weight function $w : E \to \mathbb{R}_{\geq 0}$.

[1] Previous ILP descriptions considered all possible edge pairs and only informally restricted themselves to $\mathcal{CP}$. Herein, we are more rigorous with the definition as it plays an important role when considering the polytope and its facets.

It remains to model the requirement that the realization of these variables induces a feasible—i.e., planar—planarization of the given graph. Yet, if an edge is involved in multiple crossings, the realization of the assignment $\bar{x}$ for the variable vector x is ambiguous regarding the order of the crossings on that edge: Some realizations may introduce feasible planarizations, others may not. Deciding whether there exists *some* order of the crossings that allow a feasible planarization is known as the *(strong) realizability problem*, which in turn was also shown to be NP-complete [21].

If, however, every edge is involved in at most one crossing—or if the order of the crossings on each edge is specified—the realization problem is solvable in linear time: We replace each specified crossing by a dummy node and test for planarity of the resulting graph.

Let $\mathcal{F}$ be the set of feasible (binary) solution vectors for x; i.e., each $\bar{x} \in \mathcal{F}$ allows a feasible planarization. Despite the above problems, we define the natural *crossing number polytope*

$$\mathcal{P}_{\mathrm{cr}} := \mathrm{conv}\{x \in \mathcal{F}\},$$

i.e., the convex hull of x-variable assignments that allow some feasible solution.

Realizability-based exact crossing minimization (Recm). Although it was never pointed out in any previous paper on this topic, one can define an (ad hoc rather impractical) formulation building the foundation of the two known practical formulations. Let $\bar{\mathcal{F}} := \{0,1\}^{|\mathcal{CP}|} \setminus \mathcal{F}$ be the set of infeasible binary solution vectors. For any binary vector $\bar{x}$, let $\mathbf{k}(\bar{x})$ ($\mathbf{k} \in \{\mathbf{0},\mathbf{1}\}$) be the set of edge pairs $\{e,f\}$ for which $\bar{x}_{\{e,f\}} = \mathbf{k}$. We can theoretically write the exponential number of *realizability constraints*

$$(2) \qquad \sum_{\{e,f\}\in \mathbf{1}(\bar{x})} x_{\{e,f\}} - \sum_{\{e,f\}\in \mathbf{0}(\bar{x})} x_{\{e,f\}} \le \bar{x}\bar{x}^T - 1 \qquad \forall \bar{x} \in \bar{\mathcal{F}}.$$

Since $\bar{x}\bar{x}^T$ gives the number of ones in $\bar{x}$, this constraint requires that at least one x-variable is flipped either from 1 to 0 or vice versa, relative to the forbidden vector $\bar{x}$. Since the realization problem is NP-complete, we can (unless $P = NP$) in general not construct all these constraints without testing nontrivial $\bar{x}$ vectors in exponential time to decide whether they are in $\bar{\mathcal{F}}$. This renders this *realizability-based exact crossing minimization* (Recm) formulation infeasible for practical use in general.

It is an open problem, whether there exists a practically relevant formulation using only the variables described above, while directly including the realizability subproblem. See section 9 for a discussion of Recm regarding special graph classes.

Kuratowski constraints. Kuratowski [22] gave the first classification of nonplanar graphs in 1930. Recall that H' is a *subdivision* of a graph H if it can be obtained from H by iteratively replacing edges by *chains*, i.e., simple paths internally disjoint from the rest of the graph.

Theorem 1 (Kuratowski [22]). *A finite graph is planar if and only if it does not contain a subgraph that is a* Kuratowski subdivision*, i.e., a subdivision of a K_5 or $K_{3,3}$.*

Based on this theorem, and despite the above setbacks, we can describe a special variant of the realizability constraints that forms the basis of the planarity-establishing constraints in all the subsequent formulations: A so-called *Kuratowski constraint*

$$(3) \qquad \sum_{\{e,f\}\in \mathcal{CP}(K)} x_{\{e,f\}} \ge 1$$

is conceptually bound to a specific Kuratowski subdivision K and requires at least one crossing between the edges of K. Thereby, $\mathcal{CP}(K) \subseteq E(K)^{\{2\}} \cap \mathcal{CP}$ are the edge pairs of the subdivision, where the two edges of a pair belong to nonadjacent Kuratowski paths (i.e., chains that are modeled by a single edge in the K_5 or $K_{3,3}$).

Subdivision-based exact crossing minimization (Secm) [2]. The historically first ILP crossing number formulation of practical relevance circumvents the realizability problem as follows: It does not compute the crossing number directly, but the *simple crossing number*, i.e., the smallest number of crossings under the restriction that each edge is involved in at most one crossing. Hence, there can be no ambiguity in the order of crossings. This parameter will in general be larger than $\mathrm{cr}(G)$ or might not even allow any feasible solution. Therefore, the given graph G is first expanded into a subdivision G' of G—hence the name of the formulation: Each edge is replaced by a chain of length ℓ, where ℓ is some upper bound on $\mathrm{cr}(G)$. Then, the simple crossing number $\mathrm{scr}(G') = \mathrm{cr}(G)$.

This expansion of course drastically increases the number of variables. For $e \in E$, let $E'(e)$ be the edges in G' constituting the chain corresponding to e. For clearness, we denote the variables in Secm by $z_{\{e',f'\}}$—$\forall \{e,f\} \in \mathcal{CP}, \forall e' \in E'(e), \forall f' \in E'(f)$—with the same interpretation as the x variables above.

From the polyhedral perspective, Secm hence does not consider the crossing number polytope $\mathcal{P}_{\mathrm{cr}}$ directly; instead it considers the higher-dimensional *simple crossing number polytope* $\mathcal{P}_{\mathrm{scr}}$, and uses the projection

$$\mathrm{proj}_{z \to x}: \quad \sum_{\substack{e' \in E'(e),\\ f' \in E'(f)}} z_{\{e',f'\}} = x_{\{e,f\}} \qquad \forall \{e,f\} \in \mathcal{CP}$$

to obtain a solution within $\mathcal{P}_{\mathrm{cr}}$.

Ordering-based exact crossing minimization (Oecm) [11]. While the above formulation requires up to $\Omega(|E|^4)$ variables, there exists an alternative formulation that not only requires only up to $\Omega(|E|^3)$ variables, but also works better in practice: We start by arbitrarily fixing an *orientation* of G; i.e., we assign a direction to each edge. Then we introduce additional variables $y_{e,f,g}$ for all $(e,f,g) \in E^{\langle 3 \rangle}$ with $\{e,f\}, \{e,g\} \in \mathcal{CP}$. We want $y_{e,f,g} = 1$ if and only if e is crossed by f before it is crossed by g, and 0 otherwise. Note that this also specifies that the variable can only be 1 if both crossings exist. Using these variables, we can add a linear ordering problem—hence the name of the formulation—for each edge e to the formulation.

From the polyhedral point of view, we cut a polytope $\mathcal{P}^+_{\mathrm{cr}} \supseteq \mathcal{P}_{\mathrm{cr}}$ with $|E|$ linear ordering polytopes, and obtain a feasible solution in terms of the natural variables by simply projecting out the y-variables.

Clearly, both above formulations additionally require generalized Kuratowski constraints to guarantee that the solutions are in fact feasible, i.e., that they describe a (planar) planarization.

3. Kuratowski constraints. In the following, we will show the strength of Kuratowski constraints in the context of complete and complete bipartite graphs. All above ILPs can handle weighted graphs; i.e., they can compute the weighted crossing number where a crossing between two edges costs the product of the edges' weights. We can conceptually expand any given graph $G = (V, E)$ into a complete graph by adding the edges $\bar{E} = V^{\{2\}} \setminus E$ with weights 0.

For clarity, we use $\mathcal{CP}_n$ ($\mathcal{CP}_{n,m}$) and $\mathcal{F}_n$ ($\mathcal{F}_{n,m}$) to denote $\mathcal{CP}$ and $\mathcal{F}$ (as defined above) when used in the context of a complete graph K_n (a complete bipartite graph

$K_{n,m}$, respectively). As special variants of $\mathcal{P}_{\text{cr}}$ we can then define the *complete crossing number polytope*

$$\mathcal{K}_n := \text{conv}\{x \in \mathcal{F}_n\},$$

and the *complete bipartite crossing number polytope*

$$\mathcal{K}_{n,m} := \text{conv}\{x \in \mathcal{F}_{n,m}\}.$$

As discussed above, the Kuratowski constraints are the central constraints of all known formulations. We will consider these constraints in the realm of the pure x-variable space. Furthermore, we can expand these constraints to variants where we consider more complex nonplanar graphs than only K_5 and $K_{3,3}$ subgraphs. We write V_n and E_n for the nodes and edges of the complete graph K_n, respectively. Analogously, we write V_n and V'_m for the node partitions of the graph $K_{n,m}$ with $|V_n| = n$ and $|V'_m| = m$, and $E_{n,m}$ for its edges.

DEFINITION 2 (K_5-constraint). *Fix some $n \geq 6$ and consider the complete graph $K_n = (V_n, E_n = V_n^{\{2\}})$. For each $W \in V_n^{\{5\}}$ we define a K_5-constraint C_5^W:*

$$\sum_{\substack{\{e,f\}\in\mathcal{CP}_n,\\ e,f\subset W}} x_{\{e,f\}} \geq 1.$$

DEFINITION 3 (K_m-constraint). *Fix some $n \geq 6$ and consider the complete graph $K_n = (V_n, E_n = V_n^{\{2\}})$. Assume we know $cr(K_m)^2$ for some $5 \leq m < n$; then for each $W \in V_n^{\{m\}}$ we define a K_m-constraint C_m^W as the generalization of the K_5-constraints:*

$$\sum_{\substack{\{e,f\}\in\mathcal{CP}_n,\\ e,f\subset W}} x_{\{e,f\}} \geq cr(K_m).$$

As the K_5 plays a central role in Kuratowski's classification theorem, as well as the known ILP implementation, we consider its constraints separately from its generalization to K_m-constraints. Also, the corresponding proof for this special case turns out to be simpler and easilier to follow.

It is obvious that a K_m subgraph, $m \geq 3$, can never be found in complete bipartite graphs. On the other hand, $K_{3,3}$ subgraphs occur in both complete and complete bipartite graphs. This is of particular interest since in most nonplanar real-world graphs, $K_{3,3}$ subdivisions are much easier to find (from the practical point of view) than K_5 subdivisions: Nonplanarity witnesses of planarity tests are much more likely to be subdivisions of $K_{3,3}$ rather than of K_5 [25]. Even in complete graphs K_n, there are only $\binom{n}{5}$ K_5 subgraphs, while we can enumerate $10\binom{n}{6}$ $K_{3,3}$ subgraphs.

DEFINITION 4 ($K_{3,3}$-constraint in K_n). *Fix some $n \geq 7$ and consider the complete graph $K_n = (V_n, E_n = V_n^{\{2\}})$. For each $W \in V_n^{\{3\}}$ and $W' \in (V_n \setminus W)^{\{3\}}$ we define a $K_{3,3}$-constraint $C_{3,3}^{W,W'}$:*

$$\sum_{\substack{\{e,f\}\in\mathcal{CP}_n\\ |e\cap W|=|e\cap W'|=|f\cap W|=|f\cap W'|=1}} x_{\{e,f\}} \geq 1.$$

DEFINITION 5 ($K_{3,3}$-constraint in $K_{n,m}$). *Consider some fixed $n, m \geq 3$ with $n+m \geq 7$ and consider the complete bipartite graph $K_{n,m} = (V_n \dot\cup V'_m, E_{n,m} = \{\{u, v\}$:*

[2] Currently, the crossing numbers are known for $m \leq 12$ [24].

$u \in V_n, v \in V'_m\}$). *For each* $W \in V_n^{\{3\}}$ *and* $W' \in {V'_m}^{\{3\}}$ *we define a* $K_{3,3}$-constraint $C_{3,3}^{W,W'}$*:*

$$\sum_{\substack{\{e,f\} \in \mathcal{CP}_{n,m} \\ e,f \subset (W \cup W')}} x_{\{e,f\}} \geq 1.$$

We will show the strength of the above constraints, in particular that they are *facet-defining* in the following settings:

THEOREM 6. *K_5-, K_m-, and $K_{3,3}$-constraints define facets for K_n. Furthermore, $K_{3,3}$-constraints define facets for $K_{n,m}$.*

Although the K_m-constraints include the K_5-constraints, we prove the latter separately beforehand, as it showcases the strategy used in all the subsequent proofs in the most simple way. This overall structure is described in detail in the section hereafter. The proofs themselves will then heavily reference the argumentation outlined as our proving strategy; they basically only fill in the gaps left by the overall strategy.

4. Proving strategy.

The general proof structure is as follows:

Fixing. We fix the considered input graph G and any arbitrary Kuratowski constraint C of the considered type. Such a constraint can be written as $c^T x \geq \hat{c}$. We define $X_c = \{x \in \mathcal{F} \mid c^T x = \hat{c}\}$ as the set of feasible solutions satisfying C with equality. Note that c is a binary vector; i.e., each entry in c is either 0 or 1. Let there be a valid inequality A with $a^T x \geq \hat{a}$ for which $X_c \subseteq X_a = \{x \in \mathcal{F}_n \mid a^T x = \hat{a}\}$ holds.

Aim. In order to show that C defines a facet, the remaining parts of the respective sections focus on showing that for any such inequality A we have $a_{\{e,f\}} = \lambda c_{\{e,f\}}$ and $\hat{a} = \lambda \hat{c}$ for some $\lambda > 0$. We can show this by proving that for all pairs $\{e, f\}$ with $c_{\{e,f\}} = 0$, we have $a_{\{e,f\}} = 0$; furthermore all nonzero coefficients of a have to be identical.

Partitioning. We start with partitioning the edges of the underlying complete (bipartite) graph into sets according to their incidence with W (and W'). These edge sets are denoted by $S_\star$, $\star \in \mathcal{I}$, over some suitable index set $\mathcal{I}$.

This partitioning also induces a natural partitioning of the crossing pairs in $\mathcal{CP}$, according to the memberships of their edges: A pair $\{e, f\} \in \mathcal{CP}$ belongs to the partition $P_{\star,\bullet}$ if $e \in S_\star$ and $f \in S_\bullet$, or vice versa. To avoid ambiguities, we assume that the index set $\mathcal{I}$ is *ordered* and require that $\star \leq \bullet$, using this ordering.

We will always observe that there exists exactly one index $\dagger \in \mathcal{I}$ such that all variables corresponding to $P_{\dagger,\dagger}$ have a nonzero coefficient in C. For all other partitions all induced coefficients in C are zero.

Lemma: Permutation classes. The next step of the proofs is to establish a lemma describing *permutation classes*: We prove—by the argument of permuting the index-labels of the vertices in the given complete (bipartite) graph—that all edge pairs $\{e, f\}$ belonging to the same partition $P_{\star,\bullet}$ form an equivalence class with respect to their coefficients $a_{\{e,f\}}$; i.e., they have to have a common coefficient in A, denoted by $\alpha_{\star,\bullet}$ in the following.

Theorem: Facet. The final step in each proof is performed as follows: Step by step we fix a partition $P_{\star,\bullet} \neq P_{\dagger,\dagger}$. We consider two (similar) feasible solutions—usually only differing in the routing of a single edge—that satisfy C with equality, and investigate the difference of their crossings. These solutions are chosen in such a way that they only differ in the number of crossings of type $P_{\star,\bullet}$, thus inducing $\alpha_{\star,\bullet} = 0$ in order for both solutions to satisfy A with equality—as required by the initial assumption on A.

Clearly, in later steps the considered drawings may also differ in the number of other types of crossings, if prior steps already showed that the coefficients for these crossing types are zero.

After performing this step for each permutation class $P_{\star,\bullet} \neq P_{\dagger,\dagger}$, we know that $\alpha_{\star,\bullet} = 0$ unless $\star = \bullet = \dagger$. Furthermore we know that all crossing pairs in $P_{\dagger,\dagger}$ have a common coefficient $\alpha_{\dagger,\dagger} \neq 0$. Hence the inequality A can only be a positive multiple of C. Since A is not stronger than C, each constraint of the considered type defines a facet in the crossing number polytope of the given graph.

The central ingredients in such proofs are the initial drawing scheme of the given graph that allows valid edge reroutings, and the specific edge reroutings themselves (including their order). Thereby, one has to take care not to introduce solutions where adjacent edges cross, or where a pair of edges crosses multiple times, as such solutions do not appear in crossing minimal drawings and cannot be modeled by our variables. Clearly, one has to take care that the drawing scheme is applicable to all graph sizes. In the figures below, we will usually assume that the input graph is large enough such that all possible partitions $P_{\star,\bullet}$ arise; the proofs of course also hold if some of these sets are empty as the given graph may be too small: E.g., when considering a fixed K_5 subgraph U within a K_6, each crossing involves at least one edge of U, as all the non-U edges are adjacent and therefore do not cross.

5. K_5-constraints in K_n. Fix any $n \geq 6$ and $W \in V_n^{\{5\}}$. This induces $G := K_n$, $\mathcal{F} := \mathcal{F}_n$, and a K_5-constraint $C := C_5^W$ with $\hat{c} := 1$ and

$$c_{\{e,f\}} := \begin{cases} 1 & \text{if } e, f \subset W, \\ 0 & \text{else} \end{cases} \qquad \forall \{e,f\} \in \mathcal{CP}_n.$$

We can partition all edges $e \in E_n$ using the ordered index set $\mathcal{I} = \langle 0, 1, 2 \rangle$, based on which nodes they connect:

$$e \in S_i \quad \Longleftrightarrow \quad |e \cap W| = i.$$

Considering the six induced partitions of $\mathcal{CP}_n$, we have $\dagger = 2$; i.e., $c_{\{e,f\}} = 1$ for all $\{e,f\} \in P_{2,2}$, and 0 otherwise.

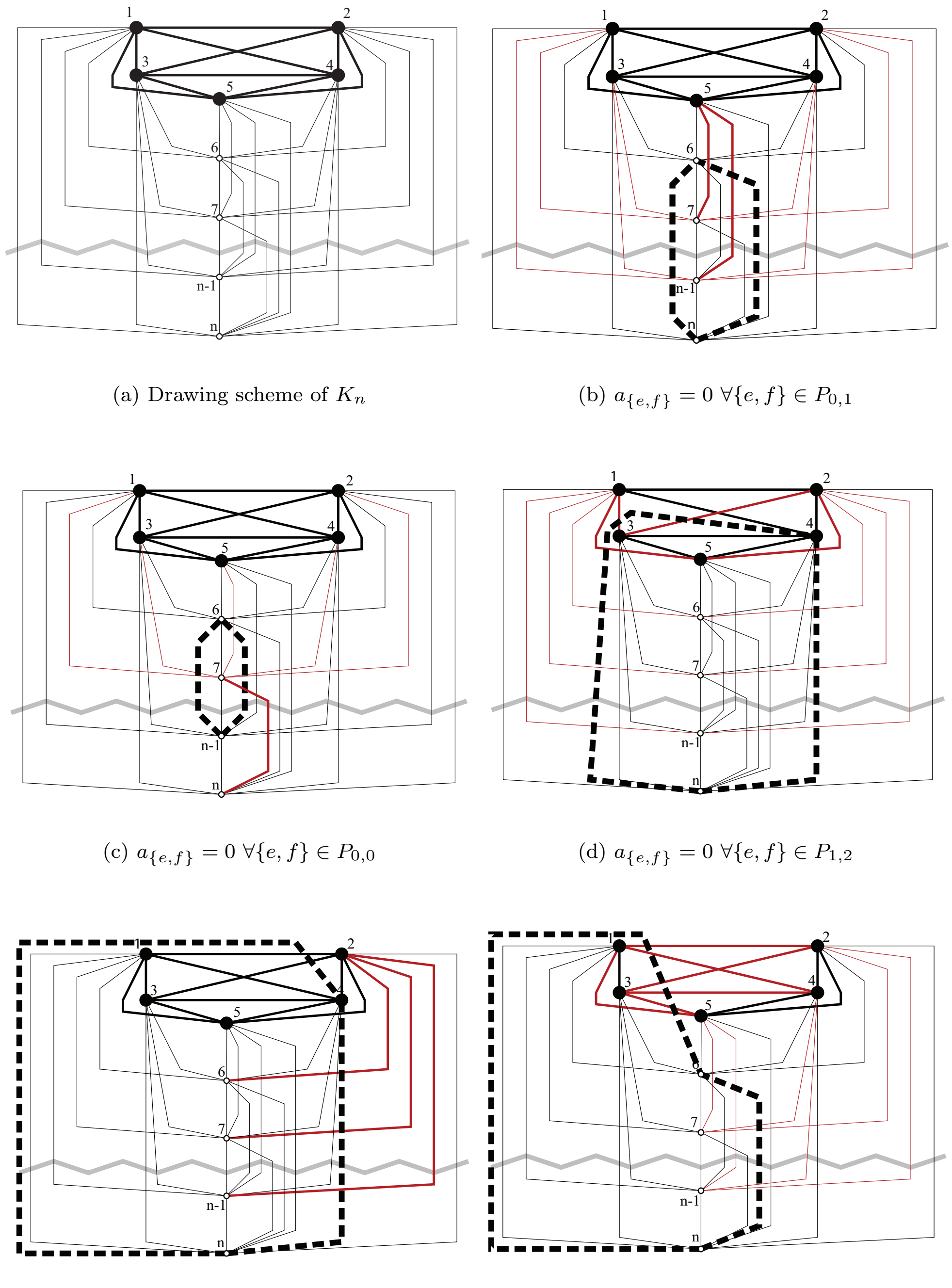
LEMMA 7. *Consider the fixing and partitioning above. For all $i, j \in \mathcal{I}$, $i \leq j$, we have*

$$\{e,f\}, \{g,h\} \in P_{i,j} \quad \Longrightarrow \quad a_{\{e,f\}} = a_{\{g,h\}}.$$

Proof. Without loss of generality (w.l.o.g.) assume that $W = \{v_1, \ldots, v_5\}$. Consider a feasible solution as shown in Figure 1(a). The edges S_2, over which C_5^W is defined, are drawn as bold lines. By expanding the drawing consistently along the gray zig-zag line, we obtain drawings for any $n \geq 6$. Clearly, this solution satisfies C_5^W with equality. Furthermore, C_5^W has exactly one variable for each possible optimal K_5 solution and only one of its variables can be 1 in any solutions of X_c.

We can relabel the nodes W arbitrarily in Figure 1(a), without violating the feasibility of the solution, nor the equality of C_5^W. We do also not violate these properties by any permutation of the node labels of $V \setminus W$.

Take any two edges e and f of the same partition S_i. Note that there is a permutation of the node labels of V and a permutation of the node labels of $V \setminus W$



(a) Drawing scheme of K_n

(b) $a_{\{e,f\}} = 0\ \forall\{e,f\} \in P_{0,1}$

(c) $a_{\{e,f\}} = 0\ \forall\{e,f\} \in P_{0,0}$

(d) $a_{\{e,f\}} = 0\ \forall\{e,f\} \in P_{1,2}$

(e) $a_{\{e,f\}} = 0\ \forall\{e,f\} \in P_{1,1}$

(f) $a_{\{e,f\}} = 0\ \forall\{e,f\} \in P_{0,2}$

FIG. 1. *K_5-constraints are facet-defining in K_n.*

in Figure 1(a), such that these two edges switch their roles. Considering all possible label permutations on W and on $V \setminus W$, A cannot distinguish between two edges if they belong to the same partition S_i for some i, because it has to satisfy all solutions arising from all possible label permutations on W and on $V \setminus W$ with equality. Since A can only differentiate between different partitions S_i, it can also only differentiate between different partitions $P_{i,j}$ of the crossing pairs. ∎

THEOREM 8. *Fix any $n \geq 6$ and $W \in V_n^{\{5\}}$. C_5^W is a facet in $\mathcal{K}_n$.*

Proof. Again, we assume w.l.o.g. that $W = \{v_1, \ldots, v_5\}$ and consider a first feasible solution as shown in Figure 1(a). We call the drawing of the path $v_5, v_6, \ldots, v_n$ the *spine* of the drawing.

$\alpha_{0,1}$: cf. Figure 1(b). We can redraw the edge $\{v_6, v_n\} \in S_0$ (bold, dashed) on the left side of the spine instead of the right side, and still obtain a solution in X_c. Thereby we only change crossings of type $P_{0,1}$, namely with $\{v_5, v_7\}, \ldots, \{v_5, v_{n-1}\}$. Since we remove more crossings then we introduce, and all coefficients of these crossings are identical, we have $\alpha_{0,1} = 0$.

$\alpha_{0,0}$: cf. Figure 1(c). We can redraw the edge $\{v_6, v_{n-1}\} \in S_0$ on the left side of the spine instead of the right side, and still obtain a solution in X_c. We remove more $P_{0,1}$ crossings than we introduce, but we already know that $\alpha_{0,1} = 0$. Hence it remains to observe that the redrawing removes a $P_{0,0}$ crossing and we therefore have $\alpha_{0,0} = 0$.

$\alpha_{1,2}$: cf. Figure 1(d). We can redraw the edge $\{v_4, v_n\} \in S_1$ on the left side of the spine instead of the right side, and let it cross through multiple edges of S_2. We still obtain a solution in X_c. We introduce more $P_{1,2}$ crossings than we remove and hence we have $\alpha_{1,2} = 0$.

$\alpha_{1,1}$: cf. Figure 1(e). We can redraw the edge $\{v_4, v_n\} \in S_1$ on the left side of the drawing instead of the right side of the spine. Thereby it crosses multiple edges of S_2 but no S_1 edges. We still obtain a solution in X_c. We already know that $\alpha_{1,2} = 0$ and can concentrate on the $P_{1,1}$ crossings. We remove such crossings without introducing any, and deduce $\alpha_{1,1} = 0$.

$\alpha_{0,2}$: cf. Figure 1(f). We can redraw the edge $\{v_6, v_n\} \in S_0$ on the left side of the drawing instead of the right side of the spine. Thereby it crosses multiple edges of S_2 but no S_1 edges. We still obtain a solution in X_c. We already know that $\alpha_{0,1} = 0$ and can concentrate on the $P_{0,2}$ crossings. We introduce such crossings without removing any, and deduce $\alpha_{0,2} = 0$. ∎

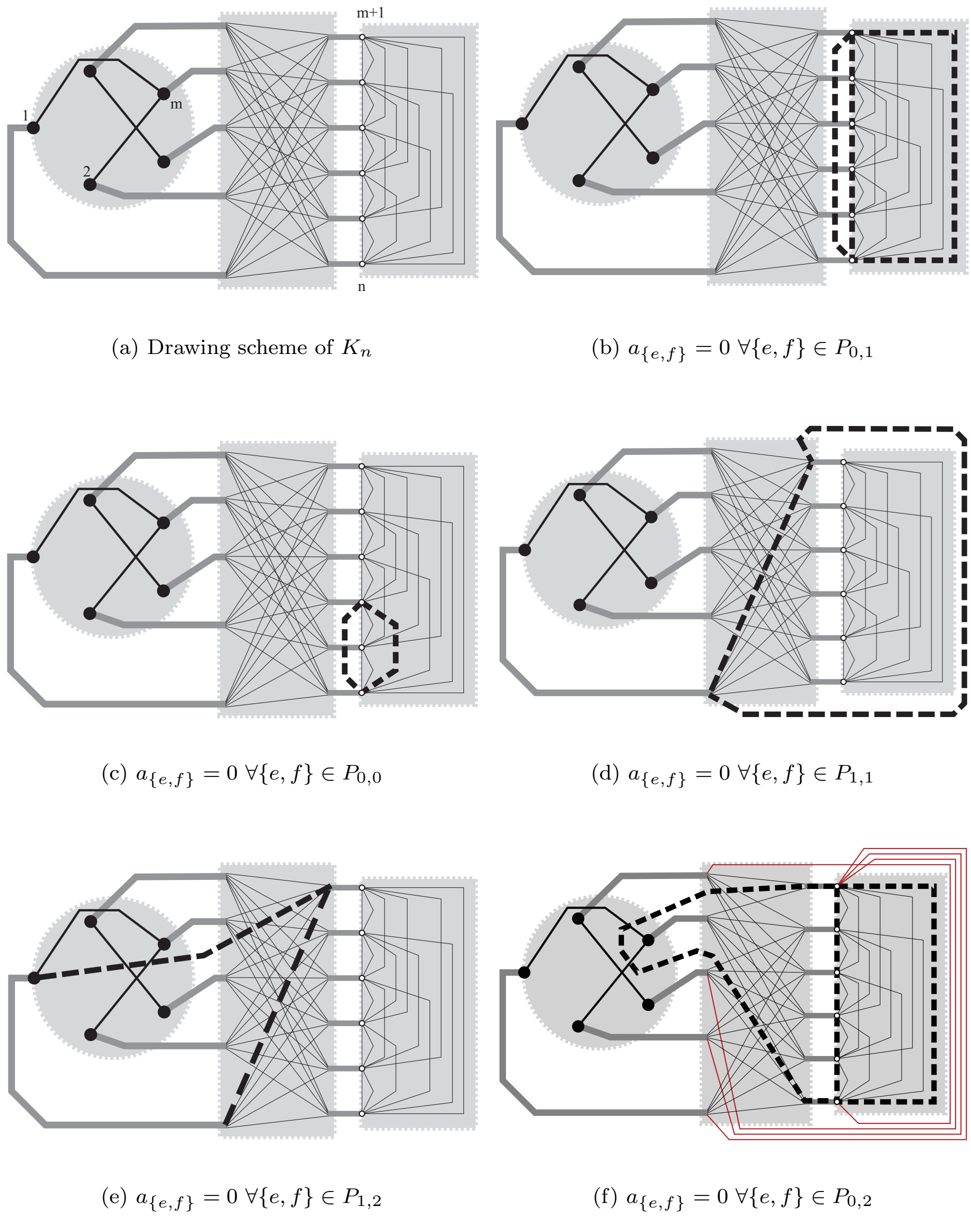
6. K_m-constraints in K_n.

Fix any $m \geq 5$, $n > m$, and $W \in V_n^{\{m\}}$. This induces $G := K_n$, $\mathcal{F} := \mathcal{F}_n$, and a K_m-constraint $C := C_m^W$ with $\hat{c} := \text{cr}(K_m)$ and (as before)

$$c_{\{e,f\}} := \begin{cases} 1 & \text{if } e, f \subset W, \\ 0 & \text{else} \end{cases} \qquad \forall \{e, f\} \in \mathcal{CP}_n.$$

We partition all edges $e \in E_n$ analogously to the previous section; i.e., we use the ordered index set $\mathcal{I} = \langle 0, 1, 2 \rangle$ and have $e \in S_i$ iff $|e \cap W| = i$. Considering the induced partitions of $\mathcal{CP}_n$, we again have $\dagger = 2$; i.e., $c_{\{e,f\}} = 1$ for all $\{e, f\} \in P_{2,2}$, and 0 otherwise.

Consider a feasible solution as shown in Figure 2(a). The nodes W lie in the cyclic shaded region on the left-hand side of the drawing. We assume that this region contains an optimal drawing of the K_m induced by W. Since we do not know the exact drawing for arbitrary m, we only visualize a couple of S_2 edges; our proof does not need the knowledge of the exact edge placements. We note that for any drawing



(a) Drawing scheme of K_n

(b) $a_{\{e,f\}} = 0 \ \forall \{e,f\} \in P_{0,1}$

(c) $a_{\{e,f\}} = 0 \ \forall \{e,f\} \in P_{0,0}$

(d) $a_{\{e,f\}} = 0 \ \forall \{e,f\} \in P_{1,1}$

(e) $a_{\{e,f\}} = 0 \ \forall \{e,f\} \in P_{1,2}$

(f) $a_{\{e,f\}} = 0 \ \forall \{e,f\} \in P_{0,2}$

FIG. 2. *K_m-constraints are facet-defining in K_n.*

of the K_m, we can choose an outer face, where at least one nodes lies on the outside of the K_m-drawing: In our drawing we assume that the left-most node lies on the outside, denoted by its placement on the circle's border; for all other nodes we neither assume that they lie on the outside, nor that they do not.

All nodes $V \setminus W$ are lined up on the left border of rectangular shaded region on the right-hand side of the drawing: The region itself contains all S_0 edges. Finally, the S_1 edges (connecting W with $V \setminus W$) are partially drawn using thick gray edges, denoting that all the considered edges are drawn in parallel without crossings, until they split up into normal thin black lines. All crossings between S_1 edges are in the shaded region at the center of the drawing.

Clearly, this solution satisfies C_m^W with equality. Using the analogous arguments as above, we can permute the node labels within W and within $V \setminus W$ and can deduce that A can only differentiate between different partitions S_i of edges and therefore only between different partitions $P_{i,j}$ of crossing pairs, giving the following:

LEMMA 9. *Consider the fixing and partitioning above. For all $i, j \in \mathcal{I}$, $i \leq j$, we have*

$$\{e, f\}, \{g, h\} \in P_{i,j} \quad \Longrightarrow \quad a_{\{e,f\}} = a_{\{g,h\}}.$$

THEOREM 10. *Fix any $m \geq 5$, $n > m$, and $W \in V_n^{\{m\}}$. C_m^W is a facet in $\mathcal{K}_n$.*

Proof. Again, we consider a first feasible solution as shown in Figure 2(a); assume w.l.o.g. that $W = \{v_1, \ldots, v_m\}$ and that v_1 is a node on the outside of the drawing induced by W. The Figures 2(b)–(f) show the necessary reroutings (and their order), required to prove the theorem. As before, the routing alternatives are emphasized as thick dashed lines. For the sake of brevity, we omit a full textual description. Such textual descriptions for this and the following proofs can be found in [6]. ◻

7. $K_{3,3}$-constraints in $K_{n,m}$. Fix any $n, m \geq 3$ with $n+m \geq 7$, $W \in V_n^{\{3\}}$, and $W' \in {V'_m}^{\{3\}}$. This induces $G := K_{n,m}$, $\mathcal{F} := \mathcal{F}_{n,m}$, and a $K_{3,3}$-constraint $C := C_{3,3}^{W,W'}$ with $\hat{c} := 1$ and

$$c_{\{e,f\}} := \begin{cases} 1 & \text{if } e, f \subset (W \cup W'), \\ 0 & \text{else} \end{cases} \qquad \forall \{e, f\} \in \mathcal{CP}_{n,m}.$$

We can partition all edges $e \in E_{n,m}$ using the ordered index set $\mathcal{I} = \langle 0, 1, 1', 2 \rangle$, based on which nodes they connect:

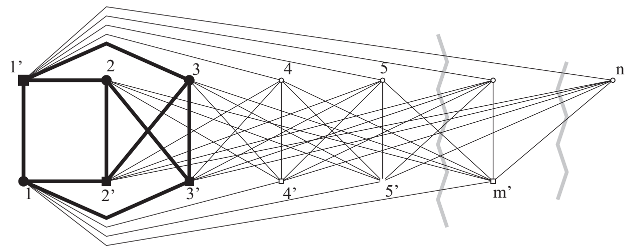
$$\begin{aligned} e \in S_0 &\Longleftrightarrow |e \cap (W \cup W')| = 0, \\ e \in S_1 &\Longleftrightarrow |e \cap W| = 1, |e \cap W'| = 0, \\ e \in S_{1'} &\Longleftrightarrow |e \cap W| = 0, |e \cap W'| = 1, \\ e \in S_2 &\Longleftrightarrow |e \cap (W \cup W')| = 2. \end{aligned}$$

Considering the 10 induced partitions of $\mathcal{CP}_{n,m}$, we have $\dagger = 2$; i.e., $c_{\{e,f\}} = 1$ for all $\{e, f\} \in P_{2,2}$, and 0 otherwise.

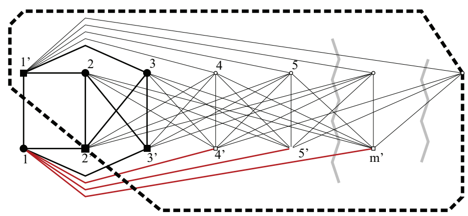
W.l.o.g. assume that $W = \{v_1, \ldots, v_3\}$ and $W' = \{v'_1, \ldots, v'_3\}$, and consider a feasible solution as shown in Figure 3(a). The edges of S_2, over which $C_{3,3}^{W,W'}$ is defined, are drawn as bold lines. Clearly, this solution satisfies $C_{3,3}^{W,W'}$ with equality. Furthermore, $C_{3,3}^{W,W'}$ has exactly one variable for each possible optimal $K_{3,3}$ solution. We can relabel the nodes W arbitrarily in this drawing, without violating the feasibility of the solution nor the equality of $C_{3,3}^{W,W'}$. The same holds for all permutations of the nodes W'. We do also not violate these properties by any permutation of the labels of the nodes of $V_n \setminus W$ and of the nodes $V'_m \setminus W'$. Analogously to above we obtain the following:

LEMMA 11. *Consider the fixing and partitioning above. For all $i, j \in \mathcal{I}$, $i \leq j$, we have*

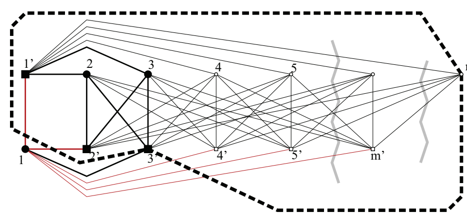
$$\{e, f\}, \{g, h\} \in P_{i,j} \quad \Longrightarrow \quad a_{\{e,f\}} = a_{\{g,h\}}.$$



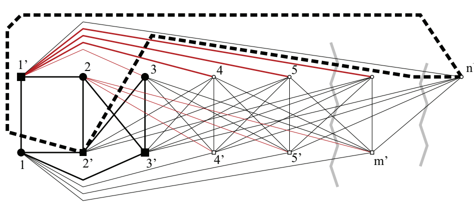
(a) Drawing scheme of $K_{n,m}$



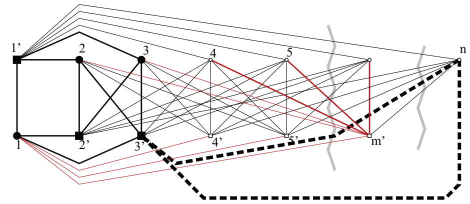
(b) $a_{\{e,f\}} = 0 \ \forall \{e,f\} \in P_{1,1'}$



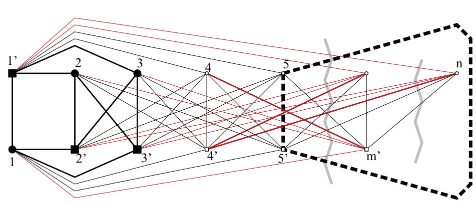
(c) $a_{\{e,f\}} = 0 \ \forall \{e,f\} \in P_{1,2} \cup P_{1',2}$



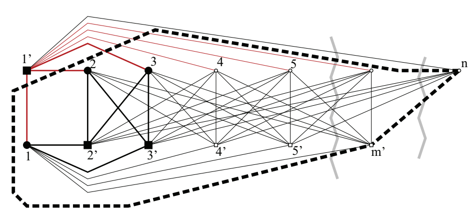
(d) $a_{\{e,f\}} = 0 \ \forall \{e,f\} \in P_{1,1} \cup P_{1',1'}$



(e) $a_{\{e,f\}} = 0 \ \forall \{e,f\} \in P_{0,1} \cup P_{0,1'}$



(f) $a_{\{e,f\}} = 0 \ \forall \{e,f\} \in P_{0,0}$



(g) $a_{\{e,f\}} = 0 \ \forall \{e,f\} \in P_{0,2}$

FIG. 3. *$K_{3,3}$-constraints are facet-defining in $K_{n,m}$.*

THEOREM 12. *Fix any $n, m \geq 3$ with $n + m \geq 7$, $W \in V_n^{\{3\}}$, and $W' \in {V'_m}^{\{3\}}$. $C_{3,3}^{W,W'}$ is a facet in $\mathcal{K}_{n,m}$.*

Proof. Figure 3 gives the initial drawing scheme and all required routing alternatives (thick dashed lines) to iteratively show that all α-coefficients are 0 except for $\alpha_{2,2}$. Note that some figures are used for two different partition sets: E.g., Figure 3(c) itself shows $\alpha_{1',2} = 0$. By symmetry (i.e., exchanging the the node labels of W and W'), we obtain $\alpha_{1,2} = 0$. Again, we omit a textual description for brevity (cf. [6]). ∎

8. $K_{3,3}$-constraints in K_n. Fix any $n > 6$ and $(W \dot\cup W') \in V_n^{\{6\}}$ with $|W| = 3$. This induces $G := K_n$, $\mathcal{F} := \mathcal{F}_n$, and a $K_{3,3}$-constraint $C := C_{3,3}^{W,W'}$ with $\hat{c} := 1$ and

$$c_{\{e,f\}} := \begin{cases} 1 & \text{if } |e \cap W| = |e \cap W'| = |f \cap W| = |f \cap W'| = 1, \\ 0 & \text{else} \end{cases} \qquad \forall \{e, f\} \in \mathcal{CP}_n.$$

We can partition all edges $e \in E_n$ using the ordered index set $\mathcal{I} = \langle 0, 1, 1', 2, 2', 2'' \rangle$, based on which nodes they connect:

$$\begin{aligned} e \in S_0 &\iff |e \cap (W \cup W')| = 0, \\ e \in S_1 &\iff |e \cap W| = 1, |e \cap W'| = 0, \\ e \in S_{1'} &\iff |e \cap W| = 0, |e \cap W'| = 1, \\ e \in S_2 &\iff |e \cap W| = 2, \\ e \in S_{2'} &\iff |e \cap W| = |e \cup W'| = 1, \\ e \in S_{2''} &\iff |e \cap W'| = 2. \end{aligned}$$

Considering the 21 induced partitions of $\mathcal{CP}_{n,m}$, we have $\dagger = 2'$; i.e., $c_{\{e,f\}} = 1$ for all $\{e, f\} \in P_{2',2'}$, and 0 otherwise. Note that the set $P_{2,2}$ ($P_{2'',2''}$) is empty, since S_2 ($S_{2''}$, resp.) forms a simple cycle of length 3, and hence all pairs of these edges are adjacent.

W.l.o.g. assume that $W = \{v_1, v_4, v_5\}$ and $W' = \{v_2, v_3, v_6\}$. Consider a feasible solution as shown in Figure 4(a). The edges of $S_{2'}$, over which $C_{3,3}^{W,W'}$ is defined, are drawn as bold lines. Clearly, this solution satisfies $C_{3,3}^{W,W'}$ with equality. Using relabeling, we again obtain the following:

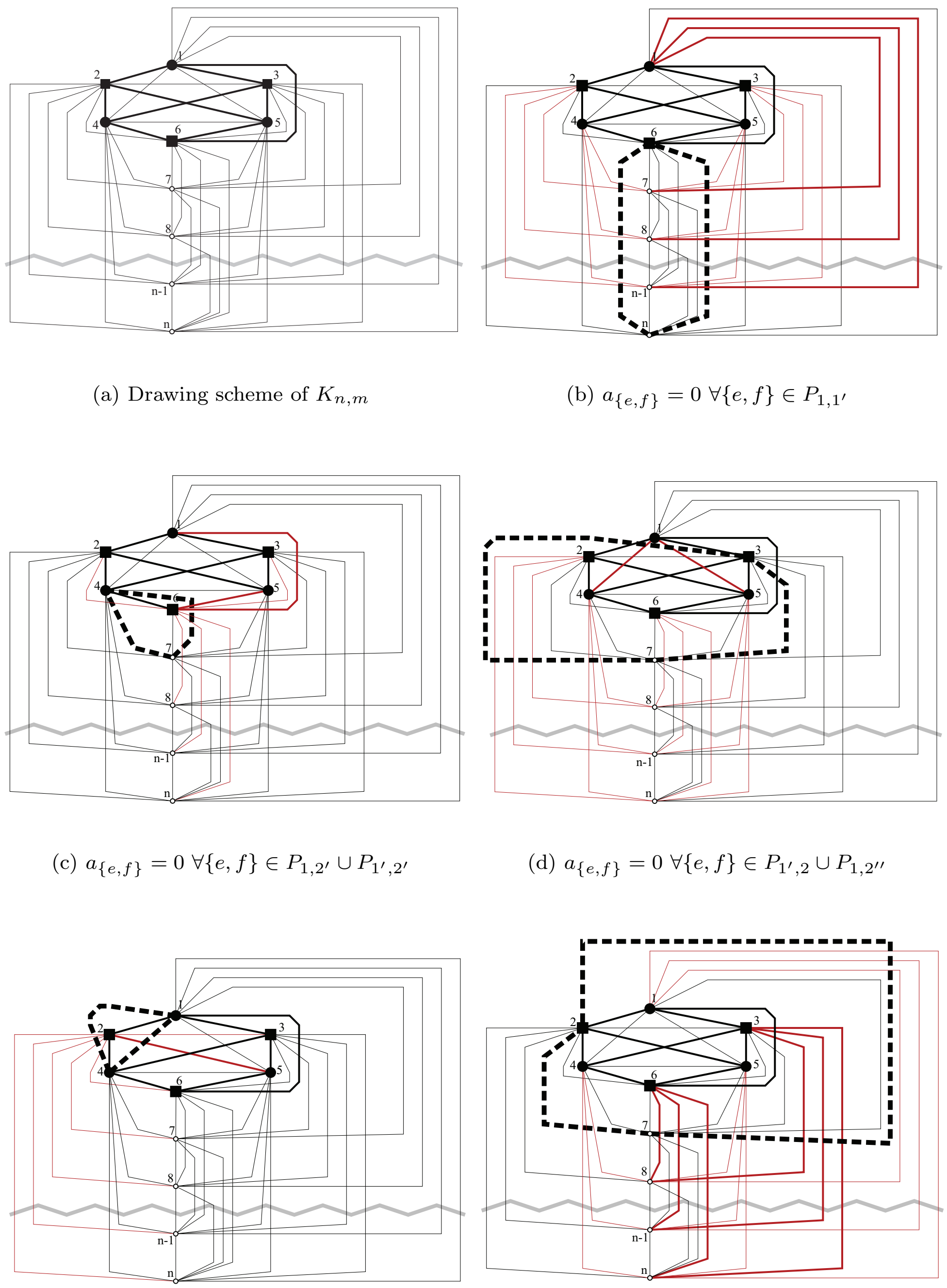
LEMMA 13. *Consider the fixing and partitioning above. For all $i, j \in \mathcal{I}$, $i \leq j$, we have*

$$\{e, f\}, \{g, h\} \in P_{i,j} \quad \Longrightarrow \quad a_{\{e,f\}} = a_{\{g,h\}}.$$

THEOREM 14. *Fix any $n \geq 7$ and $(W \dot\cup W') \in V_n^{\{6\}}$ with $|W| = 3$. $C_{3,3}^{W,W'}$ is a facet in $\mathcal{K}_n$.*

Proof. Again, we assume w.l.o.g. that $W = \{v_1, v_4, v_5\}$ and $W' = \{v_2, v_3, v_6\}$ and consider a feasible solution as shown in Figure 4(a) as our starting point. The Figure 4(b)–(l) then gives the required reroutings in order to proof the theorem. Again, a single case often settles two partition sets, based on symmetry by exchanging W and W'. ∎

This proof concludes the sequence of sections, in which we showed that the considered Kuratowski constraints indeed define facets in the complete and complete bipartite crossing number polytopes.



(a) Drawing scheme of $K_{n,m}$

(b) $a_{\{e,f\}} = 0 \ \forall \{e,f\} \in P_{1,1'}$

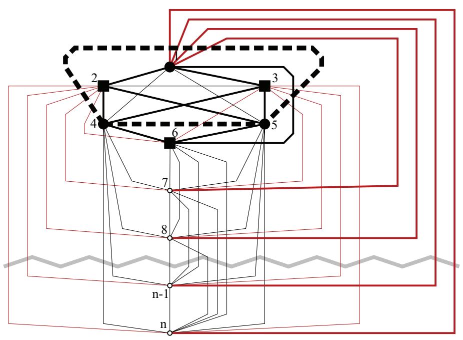
(c) $a_{\{e,f\}} = 0 \ \forall \{e,f\} \in P_{1,2'} \cup P_{1',2'}$

(d) $a_{\{e,f\}} = 0 \ \forall \{e,f\} \in P_{1',2} \cup P_{1,2''}$

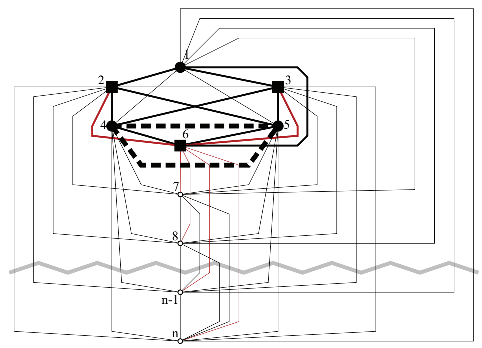
(e) $a_{\{e,f\}} = 0 \ \forall \{e,f\} \in P_{2,2'} \cup P_{2',2''}$

(f) $a_{\{e,f\}} = 0 \ \forall \{e,f\} \in P_{1,1} \cup P_{1',1'}$

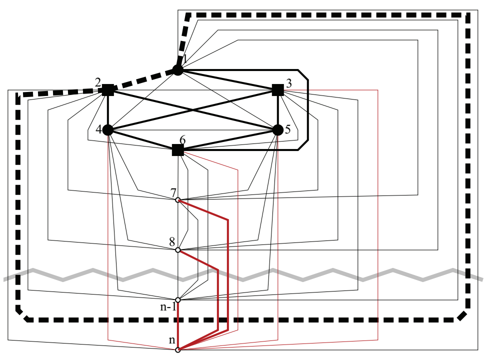
FIG. 4. *$K_{3,3}$-constraints are facet-defining in K_n (part 1).*



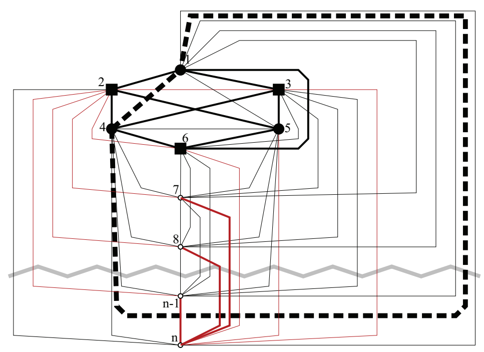
(g) $a_{\{e,f\}} = 0\ \forall\{e, f\} \in P_{1,2} \cup P_{1',2''}$



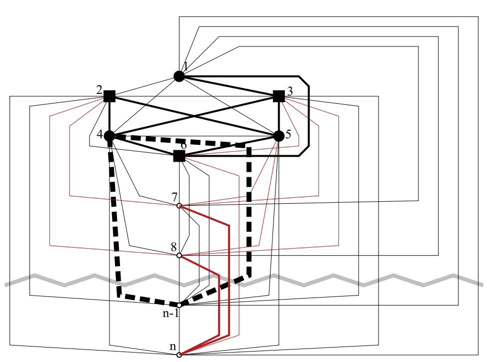
(h) $a_{\{e,f\}} = 0\ \forall\{e, f\} \in P_{2,2''}$



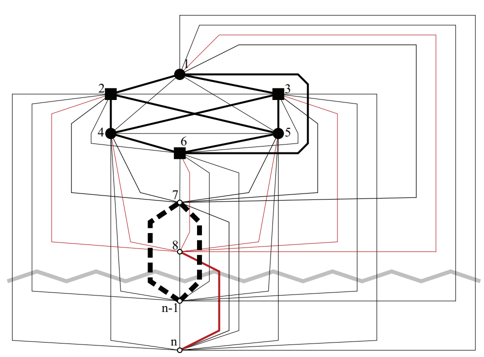
(i) $a_{\{e,f\}} = 0\ \forall\{e, f\} \in P_{0,2'}$



(j) $a_{\{e,f\}} = 0\ \forall\{e, f\} \in P_{0,2} \cup P_{0,2''}$



(k) $a_{\{e,f\}} = 0\ \forall\{e, f\} \in P_{0,1} \cup P_{0,1'}$



(l) $a_{\{e,f\}} = 0\ \forall\{e, f\} \in P_{0,0}$

FIG. 4. *$K_{3,3}$-constraints are facet-defining in K_n (part 2).*

9. Conclusions and further thoughts. To our knowledge, this paper constitutes the first polyhedral study of the crossing number problem, bringing together the topics of graph theory and algebraic discrete combinatorics. We introduced the crossing number polytope, described the known formulations in terms of it, and showed several facet-defining inequalities. Yet, we see three major open questions that we deem interesting:

General $K_{p,q}$-constraints. The probably most canonical extension of this works would be to generalize $K_{3,3}$-constraints to arbitrary $K_{p,q}$-constraints in much the same way as we generalized K_5-constraints to K_m-constraints. We would then ask whether these new constraint classes induce facets for K_n or $K_{n,m}$. It seems natural that this should hold, but constructing a corresponding proof is problematic: For the K_m-constraints, we benefit from the fact that all nodes of the K_m are interchangable. In our drawing scheme, this allows us to mask the problem that we cannot assume anything about the drawing of the K_5. For $K_{p,q}$, on the other hand, our drawing scheme would have to work for any order and interweavement of its nodes, with respect to their partition sets.

Subdivision constraints. In this paper we only considered Kuratowski constraints where the nonplanar structures appear as subgraphs. In the known formulations, we also need to consider Kuratowski constraints where these structures appear as subdivisions. Furthermore, these formulations also consider *partial planarizations*, where a subset of crossings is specified, and use Kuratowski subdivisions arising in these graphs. It may be possible to generalize the above proofs to consider cases were a specific Kuratowski edge is subdivided. Yet, the next interesting constraint classes would be arbitrary K_5 or $K_{3,3}$ subdivisions as subgraphs in G. Regarding those, it is hard to find a useful initial drawing scheme for some K_n: There is no clear way how to best draw and consider the edges connecting nodes that are only subdivision nodes for the considered Kuratowski graph.

Complete graphs. For complete graphs, the realization problem was recently shown to be solvable in polynomial time [23]. This would allow a branch-and-cut algorithm heuristically separating realizability constraints, based on rounding the solution and performing the realizability check. Due to the very specific nature of a realizability constraint—it only forbids a single infeasible solution vector instead of a whole class of them—we would only obtain a very weak and slow algorithm that is still infeasible for practical use.

The proof in [23] only gives a highly complex testing procedure, but no description in terms of forbidden minors or crossing configurations. The latter could potentially be used to construct stronger linear constraints for Recm, thus allowing a practically relevant algorithm to solve the crossing number problem for complete graphs of interesting size.

Acknowledgment. I want to thank Petra Mutzel for her introduction to and fruitful discussions on this topic.

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