SF2704 Clustering and persistence
Matrics on n point set.

Metrics.

Let X be a finite set. A *metric* on X is a function $d\colon X \times X \to \mathbf{R}$ such that:

- $d(a,b) = d(b,a)$ for any a and b in X;
- $d(a,b) \geq 0$ and $d(a,b) = 0$ if and only if $a = b$;
- $d(a,b) + d(b,c) \geq d(a,c)$ for any a, b and c in X.

We say that two sets with metrics (X,d) and (Y,d') are *isometric* if there is bijection $f\colon X \to Y$ such that $d(a,b) = d'(f(a),f(b))$ for any a and b in X.

Exercise 0.

Show that being isometric is en equivalence relation on the set of metrics on X.

We use the symbol $\mathcal{M}_X$ to denote the set of equivalence classes of the relation of being isometric on the set of metrics on X. If X has n-elements, we also denote $\mathcal{M}_X$ by $\mathcal{M}_n$

Matrices.

We use the symbol $\mathbf{M}_n$ to denote the set $n \times n$ matrices with real coefficient. We use the following symbols to denote a matrix:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{1,2} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Let S_n be the permutation group of the set of n-elements. Consider a function

$$S_n \times \mathbf{M}_n \to \mathbf{M}_n, \quad [a_{ij}] \mapsto \left[a_{\sigma(i)\sigma(j)}\right]$$

Exercise 1.

Show that the above function is an action of S_n on $\mathbf{M}_n$.

We say that two matrices $[a_{ij}]$ and $[b_{ij}]$ in $\mathbf{M}_n$ are equivalent if there is a permutation of n-elements σ such that $[b_{ij}] = \left[a_{\sigma(i)\sigma(j)}\right]$. This is an equivalence relation (prove this). The set of equivalence classes of this relation is denoted by $\mathbf{M}_n/S_n$ and called the orbit set. Its objects (the equivalence classes) are called the orbits of the action. The function $\mathbf{M}_n \to \mathbf{M}_n/S_n$ that maps a matrix $[a_{ij}]$ to its equivalence class is called the *quotient* function.

Exercise 2.

Consider two matrices in respectively $\mathbf{M}_3$ and $\mathbf{M}_4$:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 4 & 5 \\ 2 & 4 & 0 & 6 \\ 3 & 5 & 6 & 0 \end{bmatrix}$$

Descibe all the elements in the orbits of A in B of the actions of respectively S_3 and S_4 on $\mathbf{M}_3$ and $\mathbf{M}_4$.

Exercise 3.

Let:

$$\mathbf{W}_n = \left\{ [a_{ij}] \in \mathbf{M}_n \mid \begin{array}{c} a_{ii} = 0 \text{ for all } i, \\ a_{ij} = a_{ji} > 0 \text{ for all } i \neq j, \\ a_{ij} + a_{jk} \geq a_{ik} \text{ for all } i, j, \text{ and } k \end{array} \right\}$$

Prove that if $[a_{ij}]$ belongs to $\mathbf{W}_n$, then so does $\left[a_{\sigma(i)\sigma(j)}\right]$ for any permutation σ in S_n.

In this way the action of S_n on $\mathbf{M}_n$ restricts to an action of S_n on $\mathbf{W}_n$.

Exercise 4.

Prove that there is a bijection between the quotient $\mathbf{W}_n/S_n$ and the set of metrics $\mathcal{M}_n$.

Exercise 5.

Define $\mu\colon \mathbf{W}_n \times \mathbf{W}_n \to \mathbf{R}$ as follows:

$$\mu([a_{ij}], [b_{ij}]) = \min \left\{ \frac{1}{\sqrt{2}} \sqrt{\sum_{i,j} (a_{ij} - b_{\sigma(i)\sigma(j)})^2} \mid \sigma \in S_n \right\}$$

Show that the function μ factors as:

$$\mathbf{W}_n \times \mathbf{W}_n \xrightarrow{\text{quotient}} (\mathbf{W}_n/S_n) \times (\mathbf{W}_n/S_n) \xrightarrow{\bar{\mu}} \mathbf{R}, \qquad \mu = \bar{\mu} \circ \text{quotient}$$

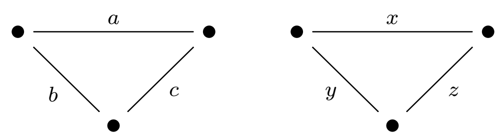
Prove that the function $\bar{\mu}\colon (\mathbf{W}_n/S_n) \times (\mathbf{W}_n/S_n) \to \mathbf{R}$ is a metric.

Exercise 6.

Use the metric $\bar{\mu}\colon (\mathbf{W}_n/S_n) \times (\mathbf{W}_n/S_n) \to \mathbf{R}$ to introduce a metric on $\mathcal{M}_n$.

Exercise 7.

Consider the following metrics on a 3 element set:



Calculate the distance (using the metric from Exercise 3) between the above metrics.