SF2729 GROUPS AND RINGS
Final Exam
Friday June 1, 2012

Time: 14:00–18:00
Allowed aids: none
Examiners: Wojciech Chachólski and Carel Faber

Present your solutions to the problems in such a way that the arguments and calculations are easy to follow. Provide detailed arguments to your answers. An answer without explanation will be given no points.

For Problem 1, the final score equals at least the number of points obtained from the homeworks of the groups part of the course. If your solution to Problem 1 is worth more points, then this will be your final score.

For Problem 2, the final score equals at least the number of points obtained from the homeworks of the rings part of the course. If your solution to Problem 2 is worth more points, then this will be your final score.

For each problem, the maximum score is 6 points.

The minimum requirements for the various grades are according to the following table:

Grade	**A**	**B**	**C**	**D**	**E**
Total credit	30	27	24	21	18

Problem 1

(6 points). List all groups of order 6 up to isomorphism and prove that these are all such groups.

Problem 2

Let R be the ring

$$\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\}.$$

It is given that R is a Euclidean domain with Euclidean multiplicative norm

$$N(a + b\sqrt{-2}) = (a + b\sqrt{-2})(a - b\sqrt{-2}) = a^2 + 2b^2.$$

a. **(2 points)**. Prove that $1 + 2\sqrt{-2}$ is not an irreducible element of R.

b. **(2 points)**. Determine a greatest common divisor in R of $2 + \sqrt{-2}$ and $4 + \sqrt{-2}$.

c. **(2 points)**. Prove that $R/\langle 3 + \sqrt{-2}\rangle$ is a finite field with 11 elements.

Problem 3

a. **(2 points)**. Let H be a subgroup of a group G. Show that if $(G : H) = 2$, then H is a normal subgroup of G.

b. **(1 point)**. Find an example of a group G and a subgroup H for which $(G : H) = 3$ and H is a normal subgroup of G.

c. **(3 points)**. Find an example of a group G and a subgroup H for which $(G : H) = 4$ and H is NOT a normal subgroup of G.

Problem 4

a. **(2 points)**. Show that $x^2 + x + 1$ is the only irreducible polynomial of degree 2 in $\mathbb{Z}_2[x]$.

b. **(2 points)**. Show that a polynomial of degree 5 in $\mathbb{Z}_2[x]$ which has no zeroes in $\mathbb{Z}_2$ and which is not divisible by $x^2 + x + 1$ is irreducible in $\mathbb{Z}_2[x]$.

c. **(2 points)**. Show that $x^5 + x^2 + 1$ is irreducible in $\mathbb{Z}_2[x]$ and determine a generator for the multiplicative group of $\mathbb{Z}_2[x]/\langle x^5 + x^2 + 1\rangle$.

Problem 5

Let S_{12} be the permutation group of the set $\{1, 2, 3, \ldots, 12\}$.

a. **(1 point)**. Is there an element τ in S_{12} for which τ^2 is odd?

b. **(2 points)**. Consider the following permutation in S_{12}:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 1 & 12 & 11 & 7 & 10 & 9 & 6 & 8 & 4 & 5 \end{pmatrix}$$

Find the cycle decomposition of σ and of σ^2. Determine if σ is odd or even.

c. **(3 points)**. Find the maximal order of a cyclic subgroup of S_{12}.

Problem 6

Let A be the set of complex numbers that are algebraic over $\mathbb{Q}$.

a. **(3 points)**. Prove that A is a subfield of $\mathbb{C}$.

b. **(3 points)**. Prove that A is algebraically closed.