KTH Teknikvetenskap

SF2729 Groups and Rings
Midterm Exam
Monday, March 15, 2010

Time: 09.00-11.00
Allowed aids: none
Examiner: Mats Boij

This midterm exam corresponds to Part I (groups part) of the final exam and the final grade will be based on the maximum of the results on this part in the midterm exam and in the final exam.

Each problem can give up to 6 points. In the first problem, you are guaranteed a minimum given by the result of the homework assignment. If you have at least 2 points from HW, you cannot get anything from Part a) of Problem 1, if you have at least 4 points from HW you cannot get anything from Part a) or Part b) of Problem 1.

The minimum requirements for the various grades are according to the following table:

Grade	**A**	**B**	**C**	**D**	**E**
Total credit	30	27	24	21	18
From Part I	13	12	11	9	8
From Part II	13	12	11	9	8

Present your solutions to the problems in a way such that arguments and calculations are easy to follow.

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(1) a) Show directly from the group axioms that the unit of a group is unique and that there are the cancellation rules

$$ab = ac \Longrightarrow b = c \quad \text{and} \quad ac = bc \Longrightarrow a = b.$$

(2)

b) Show that the symmetry group of the regular tetrahedron is isomorphic to A_4. **(2)**

c) Show that the alternating group A_n for $n \geq 3$ is generated by the 3-cycles

$$(1\,2\,3), (2\,3\,4), \ldots, (n-2\,n-1\,n).$$

(2)

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(2) a) Define what it means for a group G to act on a set X and show that such an action gives a group homomorphism $\Phi : G \longrightarrow S_X$, where S_X is the group of bijective functions from X to X under composition. **(2)**

b) Recall that the dihedral group, D_{2n}, can be presented as a factor group of the free group $F[\{r, s\}]$ with the relations $r^n = s^2 = rsrs = 1$. Let X be the set of quadratic complex polynomials $q(x)$ in one variable and let $\xi = e^{2\pi i/n}$. Show that

$$r.q(x) = \xi^{-2} q(\xi^2 x) \quad \text{and} \quad s.q(x) = x^2 q(1/x)$$

defines a well-defined action of D_{2n} on X. **(4)**

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(3) a) Show that if $\Phi : G \longrightarrow H$ is a group homomorphism and if K is a normal subgroup of G contained in $\ker \Phi$, then Φ factors through the natural quotient homomorphism $\Psi : G \longrightarrow G/K$, which sends an element of G to the coset containing it. **(2)**

b) Use the result in a) to show that the sign homomorphism, $\mathrm{sgn} : S_4 \longrightarrow \{\pm 1\}$, factors via $S_4 \longrightarrow S_3$. (*Hint*: use $K = \{\mathrm{Id}, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$.) **(4)**
