SF2729 Groups and Rings
Final exam

Monday, March 11, 2013

Examiner Tilman Bauer

Allowed aids none

Time 14:00–19:00

Present your solutions in such a way that the arguments and calculations are easy to follow. Provide detailed arguments to your answers. An answer without explanation will give few or no points.

Each problem is worth 6 points, for a total of 36 points. Your end score will be the better of the exam score and the weighted average

$$0.7\,\frac{\text{exam score}}{36} + 0.3\,\frac{\text{passed homeworks}}{14}.$$

It is thus important that you do **all problems** even if you scored high on the homework. Good luck!

Problem 1

Let G be a group. A subgroup H of a group G is a *fully invariant subgroup* if for any homomorphism $\phi : G \to G$ we have $\phi(H) \leq H$. Show that the commutator subgroup $[G, G]$ of G is a fully invariant subgroup.

Problem 2

Show that a group of order 495 cannot be simple i.e., it must have a non-trivial proper normal subgroup.

Problem 3

Let G be a group and let $x, y \in G$. Suppose that $[x, y] \in Z(G)$; show that $[x^n, y] = [x, y]^n$ for all integers $n \geq 0$.

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Problem 4

Let A be an abelian group and define a multiplication on the abelian group $R = \mathbf{Z} \times A$ by

$$(n, a) \cdot (m, b) = (nm, nb + ma).$$

1. Show that this defines a unital ring structure on $R = \mathbf{Z} \times A$ by verifying the axioms. State explicitly what the zero and unity elements are. **(3 points)**

2. Show that the group of units $R^\times$ is isomorphic to $\mathbf{Z}/2 \times A$. **(3 points)**

Problem 5

Let R be a commutative ring possessing exactly three ideals $(0) \subsetneq I \subsetneq R$.

1. Show that $I = R - R^\times$, i. e. that I consists precisely of the nonunits of R. **(4 points)**

2. Give a concrete example of such a ring. **(2 points)**

Problem 6

Let $R = \mathbf{Z}[i]$ be the ring of Gaussian integers and consider the submodule $M < R^2$ generated by the single element $(2, 1 + i)$. According to the structure theorem of finitely generated modules over PIDs, the quotient module R^2/M is isomorphic to a sum of a free module and modules of the form $R/(p^n)$, where p is a prime element. Find this decomposition and the corresponding isomorphism.