SF2729 Groups and Rings
Make-up exam

Tuesday, June 4, 2013, 08:00–13:00

Examiner Tilman Bauer

Allowed aids none

Time 14:00–19:00

Present your solutions in such a way that the arguments and calculations are easy to follow. Provide detailed arguments to your answers. An answer without explanation will give few or no points.

Each problem is worth 6 points, for a total of 36 points. Your end score will be the better of the exam score and the weighted average

$$0.7\,\frac{\text{exam score}}{36} + 0.3\,\frac{\text{passed homeworks}}{14}.$$

It is thus important that you do **all problems** even if you scored high on the homework. Good luck!

Problem 1

Let G be a group. An *automorphism* $\phi : G \to G$ is simply an isomorphism from G to itself. A subgroup $H \leq G$ is a *characteristic subgroup* if for any automorphism $\phi : G \to G$, then $\phi(H) = H$. Show that the center, $Z(G)$, of G is a characteristic subgroup.

Problem 2

Show that a group of order 1001 cannot be simple i.e., it must have a non-trivial proper normal subgroup.

Problem 3

Let G be a group and let $x, y \in G$. Suppose that $[x, y] \in Z(G)$; show that $x^n y^n = (xy)^n [x, y]^{\frac{n(n-1)}{2}}$ for all integers $n \geq 0$.

please turn over

Problem 4

Let X be a set and let $\mathcal{P}(X)$ denote the power set of X, i. e. the set of all subsets of X. For $S, T \in \mathcal{P}(X)$, define

$$S + T = (S \cup T) - (S \cap T) \quad \text{and} \quad S \cdot T = S \cap T.$$

1. Show that this defines a unital ring structure on $\mathcal{P}(X)$. State explicitly what the zero element, the unity, and the negative of an element is. **(3 points)**

2. Denote by $F(X, \mathbf{Z}/2\mathbf{Z})$ the ring of functions from X to $\mathbf{Z}/2\mathbf{Z}$, where addition and multiplication are defined by $(f + g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x)g(x)$. Show that $\mathcal{P}(X)$ and $F(X, \mathbf{Z}/2\mathbf{Z})$ are isomorphic rings. **(3 points)**

Problem 5

Show that for every $n \in \mathbf{N}$ there exists an irreducible polynomial of degree n over $\mathbf{Q}$. When using a theorem from this class, write down its full statement. **(6 points)**

Problem 6

Let A be a finitely generated abelian group. For every prime number p, the module A/pA is a vector space over $\mathbf{Z}/p\mathbf{Z}$; denote by n_p its dimension.

1. Show that if A is torsion then $n_p = 0$ for all but finitely many p. **(3 points)**

2. Show that if all n_p are the same then A is a free abelian group. **(3 points)**