SF2729 Groups and Rings
Final exam

Wednesday, March 19, 2014

Examiner Tilman Bauer

Allowed aids none

Time 14:00–19:00

Present your solutions in such a way that the arguments and calculations are easy to follow. Provide detailed arguments to your answers. An answer without explanation will give few or no points.

Each problem is worth 6 points, for a total of 36 points. Your end score will be the better of the exam score and the weighted average

$$0.75\ (\text{exam score}) + 0.25\ (\text{homework score}).$$

It is thus important that you do **all problems** even if you scored high on the homework. Good luck!

Problem 1

Show for each integer a that $35 \mid a^{13} - a$.

Problem 2

Let G be a group of order $340 = 2^2 \cdot 5 \cdot 17$.

1. Show that G has normal cyclic subgroups of orders 5 and 17. (2p)
2. Show that G has a cyclic subgroup N of order $85 = 5 \cdot 17$. (3p)
3. Show that N is normal. (1p)

Problem 3

Let G be a group such that all non-identity elements are conjugate. Show that the order of G is 1, 2, or infinite.

please turn over

Problem 4

Let R be a commutative, unital ring and $I \trianglelefteq R$ an ideal. Define

$$\sqrt{I} = \{r \in R \mid r^n \in I \text{ for some } n \geq 1\}.$$

Show that $\sqrt{I}$ is an ideal.

Problem 5

Factor the polynomial $p(x) = x^4 + x + 1$ into indecomposable factors in the following rings:

1. $\mathbf{F}_2[x]$,
2. $\mathbf{F}_3[x]$,
3. $\mathbf{Q}[x]$.

In each case, argue carefully why the factors you give are indeed indecomposable.

Problem 6

Let M be a finitely generated module over an integral domain R and let $\{x_1, \ldots, x_n\} \subseteq M$ be a maximal set of linearly independent elements and $N = \langle x_1, \ldots, x_n \rangle$ the submodule of M generated by this set. Show that M/N is a torsion module.