SF2729 Groups and Rings
Make-up exam

Wednesday, May 21, 2014

Examiner Tilman Bauer

Allowed aids none

Time 14:00–19:00

Present your solutions in such a way that the arguments and calculations are easy to follow. Provide detailed arguments to your answers. An answer without explanation will give few or no points.

Each problem is worth 6 points, for a total of 36 points. Your end score will be the better of the exam score and the weighted average

$$0.75\ (\text{exam score}) + 0.25\ (\text{homework score}).$$

It is thus important that you do **all problems** even if you scored high on the homework. Good luck!

Problem 1

Let G be a group with an element x such that $xyx = y^3$ for all $y \in G$. Show that

1. $x^2 = e$ (1p);
2. $y^8 = e$ for all $y \in G$ (5p).

Solution

For $y = e$ we get that $x^2 = xex = e^3 = e$. From this we get

$$y = x^2yx^2 = x(xyx)x = xy^3x = (y^3)^3 = y^9.$$

Dividing by y gives $y^8 = 1$.

Problem 2

Let G be a simple group of order $168 = 2^3 \cdot 3 \cdot 7$ (i. e. a group with no nontrivial normal subgroups). How many elements of order 7 does G have?

Solution

By the Sylow theorems, $n_7 \equiv 1 \pmod 7$ and $n_7 \mid 24$. Since the group is simple, $n_7 \neq 1$ (otherwise the Sylow 7-subgroup would be normal), so $n_7 = 8$ is the only possibility. All eight Sylow 7-subgroups intersect trivially, and every element of order 7 is contained in some Sylow 7-subgroup, so the number of elements of order 7 is

$$8 \cdot (7 - 1) = 48.$$

Problem 3

Let G be a finite group such that p is the smallest prime divisor of $|G|$, and let H be a subgroup of index p. Show that H is normal. You can (but do not have to) follow the following outline of a proof:

1. Define a homomorphism $\phi\colon G \to S_p$, the symmetric group on p letters, using the action of G on the set G/H, and show that the $|\operatorname{im}(\phi)|$ divides $|G|$. (2p)
2. Show that $|\operatorname{im}(\phi)| = p$. (2p)
3. Show that $H = \ker(\phi)$, and thus H is a normal subgroup. (2p)

Solution

The group G acts on the set G/H by left multiplication. Thus we obtain a homomorphism $\phi\colon G \to S_p$, where S_p is identified with the group of bijections of G/H. Thus the order of the image of ϕ divides $p!$. But the order of the image is isomorphic to the index of the stabilizer and thus also divides the order of G. Thus $|\phi(G)| \mid \gcd(p!, |G|) = p$. If $|\phi(G)| = 1$ then $G = H$, which cannot happen. Thus $|\phi(G)| = p$. Since $H \subseteq \ker(\phi)$, we must have that $H = \ker(\phi)$ because both subgroups of G have index p. Thus H is normal.

Problem 4

Let k be a field and consider the ring $R = k[x]/(x^2 - 1)$.

1. Show that the ring R is isomorphic with $k[y]/(y^2)$ if $2 = 0$ in k. (3p)
2. Show that the ring R is isomorphic with $k \times k$ if $2 \neq 0$ in k. (3p)

Solution

If $2 = 0$ in k then $x^2 - 1 = (x-1)^2$, thus the map $k[y]/(y^2) \to k[x]/(x^2 - 1)$ sending $a + by$ to $a + b(x - 1)$ is a ring isomorphism.

If $2 \neq 0$, define a map $\phi\colon R \to k \times k$ by $\phi([p]) = (p(1), p(-1))$ for $p \in k[x]$. This is well-defined since 1 and -1 are both zeroes of $x^2 - 1$. As a product of evaluation homomorphisms, ϕ is a ring homomorphism. To show it is injective, assume $p(1) = p(-1) = 0$. Thus $(x-1)(x+1) \mid p$, hence $[p] = 0 \in k[x]/(x^2-1)$. For surjectivity, note that

$$\phi(\frac{a+b}{2} + \frac{a-b}{2}x) = (a,b),$$

using that $2 \neq 0$ in k.

Problem 5

Compute $\gcd(7-4\sqrt{d}, 8-\sqrt{d})$ in the ring $\mathbf{Z}[\sqrt{d}]$ for $d = -1$ and $d = -2$. (3p each)

Solution

The Euclidean algorithm for $d = -1$ gives $8 - i = (7-4i) + (1+3i), 7 - 4i = -2i(1+3i) + 1 - 2i$, and a gcd is $1 - 2i$.

For $\gcd(7-4\sqrt{-2}, 8-\sqrt{-2})$, we perform the first step of the Euclidean algorithm, $7 - 4\sqrt{-2} = 1 \cdot (8-\sqrt{-2}) - 1 - 3\sqrt{-2}$. The remainder has norm $1^2 + 2 \cdot 3^2 = 19$, whereas $N(8-\sqrt{-2}) = 8^2 + 2 = 66$. Since the norms are coprime, so are the numbers, hence the gcd is 1.

Problem 6

Let R be a principal ideal domain which is not a field, and M a finitely generated R-module. Show that for every $x \in M - \{0\}$ there is an $r \in R - \{0\}$ such that x is not divisible by r, i. e. there is no $y \in M$ such that $ry = x$.

Solution

The statement is trivially true for $M = \{0\}$. Otherwise, by the structure theorem, M is isomorphic to a direct sum of a free module R^n and torsion modules of the from $R/(p^k)$ for various primes p and natural numbers k, and it suffices to consider the cases $M = R$ and $M = R/(p^k)$. If $M = R/(p^k)$ then no $x \in M - \{0\}$ is divisible by p^k. If on the other hand $M = R$, $x \in M - \{0\}$, choose a prime element $p \in R$ (which exists since R is not a field) and write $x = p^a z$ with $p \nmid z$, which is uniquely possible because R is a UFD. Then x is not divisible by p^{a+1} since that would imply $p \mid z$.