SF2729 Groups and Rings
Make-up exam

Wednesday, May 21, 2014

Examiner Tilman Bauer

Allowed aids none

Time 14:00–19:00

Present your solutions in such a way that the arguments and calculations are easy to follow. Provide detailed arguments to your answers. An answer without explanation will give few or no points.

Each problem is worth 6 points, for a total of 36 points. Your end score will be the better of the exam score and the weighted average

$$0.75\ (\text{exam score}) + 0.25\ (\text{homework score}).$$

It is thus important that you do **all problems** even if you scored high on the homework. Good luck!

Problem 1

Let G be a group with an element x such that $xyx = y^3$ for all $y \in G$. Show that

1. $x^2 = e$ (1p);
2. $y^8 = e$ for all $y \in G$ (5p).

Problem 2

Let G be a simple group of order $168 = 2^3 \cdot 3 \cdot 7$ (i. e. a group with no nontrivial normal subgroups). How many elements of order 7 does G have?

please turn over

Problem 3

Let G be a finite group such that p is the smallest prime divisor of $|G|$, and let H be a subgroup of index p. Show that H is normal. You can (but do not have to) follow the following outline of a proof:

1. Define a homomorphism $\phi\colon G \to S_p$, the symmetric group on p letters, using the action of G on the set G/H, and show that the $|\operatorname{im}(\phi)|$ divides $|G|$. (2p)
2. Show that $|\operatorname{im}(\phi)| = p$. (2p)
3. Show that $H = \ker(\phi)$, and thus H is a normal subgroup. (2p)

Problem 4

Let k be a field and consider the ring $R = k[x]/(x^2 - 1)$.

1. Show that the ring R is isomorphic with $k[y]/(y^2)$ if $2 = 0$ in k. (3p)
2. Show that the ring R is isomorphic with $k \times k$ if $2 \neq 0$ in k. (3p)

Problem 5

Compute $\gcd(7 - 4\sqrt{d}, 8 - \sqrt{d})$ in the ring $\mathbf{Z}[\sqrt{d}]$ for $d = -1$ and $d = -2$. (3p each)

Problem 6

Let R be a principal ideal domain which is not a field, and M a finitely generated R-module. Show that for every $x \in M - \{0\}$ there is an $r \in R - \{0\}$ such that x is not divisible by r, i. e. there is no $y \in M$ such that $ry = x$.