SF2729 Groups and Rings
Solution to Problem 2.3

Problem 3.. Show that the symmetric group S_n has a presentation of the form

(1) $\quad S_n = \langle a_1, \dots, a_{n-1} \mid a_i^2 = \mathrm{id},\ a_i a_j = a_j a_i \text{ for } |i-j| > 1,\ (a_i a_{i+1})^3 = \mathrm{id}\rangle$

Hint: Bubble sort.

Let a_i be the two-cycle $(i\ \ i+1)$ for $i = 1, \dots, n-1$.

The relations are satisfied: We have that $a_i^2 = (i\ \ i+1)^2 = \mathrm{id}$ for any i. Furthermore, a_i and a_j are disjunct 2-cycles if $|i-j| > 1$, hence they commute in this case. Lastly,

$$(a_i a_{i+1})^3 = \big((i\ \ i+1)(i+1\ \ i+2)\big)^3 = (i\ \ i+1\ \ i+2)^3 = \mathrm{id},$$

showing that the relations are indeed satisfied.

The a_i generate S_n: We prove this by induction in n, the case $n = 1$ being trivial. Let $\sigma \in S_n$ be a permutation with $\sigma(n) = k$. If $k = n$ then $\sigma \in S_{n-1}$ and we are done by induction. If $k < n$ then the permutation

$$\sigma' = (n-1\ \ n)(n-2\ \ n-1)\cdots(k\ \ k+1)\sigma = a_{n-1}\cdots a_k\sigma$$

has $\sigma' \in S_{n-1}$, hence is a product of a_i by induction. Thus so is

$$\sigma = a_k a_{k+1} \cdots a_{n-1}\sigma'.$$

The list of relations is complete: It is in theory possible that there are additional relations in S_n we cannot derive from the three types of relations given in the problem, and we have to argue that this is not the case. This is somewhat harder and I did not expect you to give a proof, but here is one.

Let G_n be the group generated as in (1). We have a homomorphism $G_n \to S_n$ sending a_k to $(k\ \ k+1)$, which is surjective by what we've shown above. To see it is also injective, we prove that $|G_n| = n!$, which of course is also the cardinality of the symmetric group. We proceed by induction in n, the case $n = 1$ being clear.

Consider the set of left cosets $X = G_n/G_{n-1}$. Any element is of the form gG_{n-1} for some $g \in G_n$; in fact we will show it is of the form

$$a_k a_{k+1} \dots a_{n-1} G_{n-1} \quad \text{or just} \quad G_{n-1}.$$

Any element $g \in G_n$ can be written as some string $a_{i_1} \cdots a_{i_r}$ with $i_j \neq i_{j+1}$. Since $a_1, \dots, a_{n-2} \in G_{n-1}$, we can assume that $i_r = n-1$, otherwise we can absorb a_{i_r} into G_{n-1}. Now let j be the largest index such that $i_j \neq i_{j+1} - 1$. If $i_j < i_{j+1} - 1$, we can apply the commutation relation to move a_{i_j} to the right and into G_{n-1}. If $i_j > i_{j+1} + 1$ for some j, we similarly can move a_{i_j} to the right. Finally, if $i_j = i_{j+1} + 1$, we have a substring $a_{i_j} a_{i_j - 1} a_{i_j}$ to which we can apply the third relation to replace it by $a_{i_j-1} a_{i_j} a_{i_j-1}$. Then we can move the right hand side a_{i_j-1} into G_{n-1} and continue normalizing.

Since all the cosets $a_k a_{k+1} \dots a_{n-1} G_{n-1}$ are different and different from G_{n-1}, there are a total of n of them. Since all cosets have the same size, which by induction is $(n-1)!$, and their disjoint union is G_n, we have therefore seen that $|G_n| = n!$. $\square$