# Algebra and geometry through projective spaces

# Problems on the topology of projective spaces

due: Tuesday, February 17

This is a set of problems on so-called *weighted projective spaces*, a generalization of complex projective space $\mathbf{C}P^n$.

Given a sequence $\chi = (\chi_0, \dots, \chi_n)$ of natural numbers (excluding 0), define a space $\mathbf{C}P(\chi)$ by

$$\mathbf{C}P(\chi) = (\mathbf{C}^{n+1} - \{0\})/\sim,$$

where $(z_0, \dots, z_n) \sim (\lambda^{\chi_0} z_0, \dots, \lambda^{\chi_n} z_n)$ for all $\lambda \in \mathbf{C}^\times$. Thus $\mathbf{C}P(1, 1, \dots, 1) \cong \mathbf{C}P^n$.

**Problem 1.** Show that $\mathbf{C}P(\chi)$ is a second countable compact Hausdorff space.

**Problem 2.** Show that $\mathbf{C}P(\chi)$ is homeomorphic to the quotient of $\mathbf{C}P^n$ by an action of a finite abelian group. What is that group and its action?

**Problem 3.** Show that the spaces $\mathbf{C}P(\chi)$ and $\mathbf{C}P(k\chi)$ are homeomorphic for any $k \geq 1$. By $k\chi$ we mean, of course, $(k\chi_0, \dots, k\chi_n)$.

**Problem 4.** Let $p$ be a prime. Show that the map

$$\mathbf{C}P(\chi_0, p\chi_1, p\chi_2, \dots, p\chi_n) \xrightarrow{[x_0:\cdots:x_n] \mapsto [x_0^p:x_1:\cdots:x_n]} \mathbf{C}P(\chi)$$

is a homeomorphism.

*Remark:* One can therefore write any weighted projective space with a weighting $\chi$ such that every prime $p$ does not divide at least two weights. One can show that two weighted projective spaces are homeomorphic if and only if their weights, normalized in this way, agree, but this is beyond the scope of these exercises.

**Problem 5.** Show that there are weights $\chi$ such that $\mathbf{C}P(\chi)$ is not a manifold.