

# Chapter 1

## Topological spaces

### 1.1 The definition of a topological space

One of the first definitions of any course in calculus is that of a continuous function:

**Definition 1.1.1.** *A map  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous at the point  $x_0 \in \mathbb{R}$  if for every  $\epsilon > 0$  there is a  $\delta > 0$  such that if  $|x - x_0| < \delta$  then  $|f(x) - f(x_0)| < \epsilon$ . It is simply called continuous if it is continuous at every point.*

This  $\epsilon$ - $\delta$ -definition is tailored to functions between the real numbers; the aim of this section is to find an abstract definition of continuity.

As a first reformulation of the definition of continuity, we can say that  $f$  is continuous at  $x \in \mathbb{R}$  if for every open interval  $V$  containing  $f(x)$  there is an open interval  $U$  containing  $x$  such that  $f(U) \subseteq V$ , or equivalently  $U \subseteq f^{-1}(V)$ . The original definition would require  $x$  and  $f(x)$  to be at the centers of the respective open intervals, but as we can always shrink  $V$  and  $U$ , this does not change the notion of continuity.

We can even do without explicitly mentioning intervals. Let us call a subset  $U \subseteq \mathbb{R}$  open if for every  $x \in U$  there,  $U$  contains an open interval containing  $x$ . We now see that a  $f$  is continuous everywhere if and only if  $f^{-1}(V)$  is open whenever  $V$  is.

Thus by just referring to open subsets of  $\mathbb{R}$ , we can define continuity. We are now in a place to make this notion abstract.

**Definition 1.1.2.** *Let  $X$  be a set. A topology on  $X$  is a collection  $\mathcal{U} \subseteq \mathcal{P}(X)$  of subsets of  $X$ , called the open sets, such that:*

(Top1)  $\emptyset \in \mathcal{U}$  and  $X \in \mathcal{U}$ ;

**(Top2)** If  $X, Y \in \mathcal{U}$  then  $X \cap Y \in \mathcal{U}$ ;

**(Top3)** If  $\mathcal{V} \subseteq \mathcal{U}$  is an arbitrary subset of open sets then  $\bigcup \mathcal{V} = \bigcup_{V \in \mathcal{V}} V \in \mathcal{U}$ .

A space  $X$  together with a topology  $\mathcal{U}$  is called a topological space; we will usually abuse notation and say that  $X$  is a topological space without mentioning  $\mathcal{U}$ .

A function  $f: X \rightarrow Y$  between topological spaces is called continuous if preimages of open sets under  $f$  are open.

Confusingly, a *closed* subset  $A \subseteq X$  is not one that is not open but rather one such that the complement  $X - A$  is open. Thus there are many subsets that are neither open nor closed (for instance, half-open intervals in  $\mathbb{R}$  or the subset  $\mathbb{Q} \subset \mathbb{R}$ ) and a few that are both (for instance, the empty set).

Since the preimage of the complement of a set under a map is the complement of the preimage, we might equally well define a continuous map to be one such that preimages of closed sets are closed.

*Example 1.1.3.* The standard topology on the space  $\mathbb{R}^n$  has as open sets those sets  $U \subseteq \mathbb{R}^n$  such that for every  $x_0 \in U$  there is an  $\epsilon > 0$  such that

$$B_\epsilon(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < \epsilon\} \subseteq U$$

**Lemma 1.1.4.** A set  $A \subseteq \mathbb{R}^n$  is closed if and only if for every convergent sequence  $x_i \in A$ ,  $x = \lim_{i \rightarrow \infty} x_i \in A$ .

*Proof.* We show the “if” direction first. Let  $A \subseteq \mathbb{R}^n$  be a subset with the convergent sequence property, and assume  $A$  is not closed, i. e.,  $\mathbb{R}^n - A$  is not open. Then there exists a point  $x \in \mathbb{R}^n - A$  such that for every  $\epsilon = \frac{1}{n}$  there is a point  $x_n \in B_\epsilon(x)$  which belongs to  $A$ . But this sequence converges to  $x$ , which therefore has to lie in  $A$ , contrary to our assumption.

Conversely, assume  $\mathbb{R}^n - A$  is open and  $(x_i)$  is a convergent sequence in  $A$  with limit  $x$ . If it was true that  $x \notin A$  then there would be some  $\epsilon > 0$  such that  $B_\epsilon(x) \subseteq \mathbb{R}^n - A$ . Since  $x_i$  converges to  $x$ , almost all  $x_i$  would have to lie in  $\mathbb{R}^n - A$ , contrary to our assumption.  $\square$

*Example 1.1.5.* Any set  $X$  can be given two extreme topologies: one where all subsets are open (this is called the *discrete topology*) and one where no subsets except  $\emptyset$  and  $X$  itself are open (this is called the *indiscrete* or *trivial topology*).

**Definition 1.1.6.** A homeomorphism between two topological spaces  $X$  and  $Y$  is a bijective, continuous map  $f: X \rightarrow Y$  whose inverse  $f^{-1}$  is also continuous.

## 1.2 Subspaces, quotient spaces, and product spaces

Let  $X$  be a topological space and  $Y \subset X$  a subset. Then  $Y$  inherits a topology by defining the open sets of  $Y$  to be the intersections  $U \cap Y$ , where  $U$  is open in  $X$ . This topology is called the *subspace topology*. Note that in the case where  $Y$  is not itself open, open sets in  $Y$  are not necessarily open in  $X$ .

*Example 1.2.1.* Let  $X = \mathbb{R}$  and  $Y = [0, 1] \subseteq \mathbb{R}$  be the closed unit interval. Then the intervals  $I_1 = (\frac{1}{3}, \frac{2}{3})$  and  $I_2 = (\frac{2}{3}, 1]$  are both open in  $Y$  because

$$I_1 = (\frac{1}{3}, \frac{2}{3}) \cap Y \quad \text{and} \quad I_2 = (\frac{2}{3}, \frac{4}{3}) \cap Y.$$

However, only  $I_1$  is open in  $X$  because the point  $1 \in I_2$  is not contained in an open interval which itself is contained in  $I_2$ .

Now let  $X$  be a topological space and let  $\sim$  be an equivalence relation on  $X$ . There is a natural surjective map  $p: X \rightarrow X/\sim$  to the set of equivalence classes under the relation  $\sim$ , sending a point  $x$  to its equivalence class  $[x]$ . Again, we can define an induced topology on  $X/\sim$  by decreeing that a subset  $V \subseteq X/\sim$  is open if and only if its preimage  $p^{-1}(V)$  is open in  $X$ .

It may be worthwhile to stop to verify that this indeed defines a topology. For (Top1), note that  $p^{-1}(\emptyset) = \emptyset$  and  $p^{-1}(X/\sim) = X$ , so those two extreme subsets are open. Next, if  $V_1$  and  $V_2 \subseteq X/\sim$  are open then  $p^{-1}(V_1 \cap V_2) = p^{-1}(V_1) \cap p^{-1}(V_2)$ , which is open in  $X$  by (Top2), thus so is  $V_1 \cap V_2$  in  $X/\sim$ . Lastly, if  $\mathcal{V}$  is a family of open sets in  $X/\sim$  then  $p^{-1}(\bigcup \mathcal{V}) = \bigcup_{V \in \mathcal{V}} p^{-1}(V)$  is open in  $X$ , proving (Top3).

A special case of the quotient space construction, which is of particular interest, is the quotient space with respect to a group action. Let  $G$  be a group acting on a topological space  $X$ . We can define an equivalence relation  $\sim_G$  on  $X$  by  $x \sim_G y$  if and only if there exists some  $g \in G$  such that  $g.x = y$ . In this way, the quotient space  $X/G$  of equivalence classes, or  $G$ -orbits, becomes a topological space.

Quotient spaces can be quite ill-behaved for random equivalence relations. In practice, one wishes the projection map  $p: X \rightarrow X/\sim$  to not only be continuous but also *open*, i. e. images of open subsets of  $X$  are open in  $X/\sim$ . Luckily, this is always the case for group actions:

**Lemma 1.2.2.** *Let  $G$  be a group acting continuously on a topological space  $X$ . Then the projection map  $p: X \rightarrow X/G$  is open.*

*Proof.* Let  $U \subseteq X$  be open. We need to show that  $p(U)$  is open, i. e. by the definition of quotient topology, that  $p^{-1}(p(U))$  is open in  $X$ . Note that the translates  $gU$  of  $U$  are all open because they are the inverse images of  $U$  under the continuous map  $X \xrightarrow{g^{-1}} X$ . Thus

$$p^{-1}(p(U)) = \bigcup_{g \in G} gU,$$

as a union of open sets, is open. □

Given two topological spaces  $X$  and  $Y$ , the product  $X \times Y$  becomes a topological space with the *product topology* which is defined as follows: a subset  $U \subseteq X \times Y$  is open iff it is an arbitrary union of sets of the form  $U \times V$ , where  $U$  is open in  $X$  and  $V$  is open in  $Y$ .

## 1.3 Separation

The indiscrete topology  $\{\emptyset, X\}$ , defined on any space  $X$ , is not a very useful one because geometrically, all points are clumped together – they cannot be *separated* from each other. More precisely, there are no nonconstant continuous maps  $X \rightarrow Y$  for any “reasonable” space  $Y$  such as the real line.

**Definition 1.3.1.** *A topological space  $X$  is Hausdorff if for any choice of two distinct points  $x, y \in X$  there are disjoint open sets  $U, V$  in  $X$  such that  $x \in U$  and  $y \in V$ .*

The indiscrete topology is manifestly not Hausdorff unless  $X$  is a singleton. The standard topology on  $\mathbb{R}^n$  is Hausdorff: for  $x \neq y \in \mathbb{R}^n$ , let  $d$  be half the Euclidean distance between  $x$  and  $y$ . Then  $U = B_d(x)$  and  $V = B_d(y)$ , the open balls of radius  $d$  centered at  $x$  resp.  $y$ , fulfill the requirements.

A convenient alternative way to define Hausdorffness is as follows:

**Lemma 1.3.2.** *Let  $X$  be a topological space and  $\Delta \subseteq X \times X$  the diagonal, i. e. the set  $\Delta = \{(x, x) \in X \times X\}$ . Then  $X$  is Hausdorff if and only if  $\Delta$  is closed in  $X \times X$ .*

*Proof.* Let us first assume  $\Delta$  is closed. Let  $x, y$  be two distinct points in  $X$ . Then  $(x, y)$  lies in the open set  $X \times X - \Delta$ . By the definition of the product topology, there are open sets  $U, V$  containing  $x$  and  $y$ , respectively, such that  $U \times V \subseteq X \times X - \Delta$ . Thus  $U$  and  $V$  are disjoint open sets separating  $x$  and  $y$ .

Conversely, assume  $X$  is Hausdorff. It suffices to produce an open set  $U \times V$  for every point  $(x, y) \in X \times X - \Delta$  such that  $x \in U$ ,  $y \in V$ , and such that  $U \times V \subseteq X \times X - \Delta$ . Any two separating open sets  $U$  and  $V$  of  $x$  and  $y$  will work for this, and they exist because  $X$  is Hausdorff.  $\square$

Hausdorffness is inherited by subspaces, but not necessarily by quotient spaces. However, we have:

**Lemma 1.3.3.** *Let  $X/\sim$  be a quotient space of a Hausdorff space  $X$  by an equivalence relation  $\sim$  such that the projection map  $p: X \rightarrow X/\sim$  is open. Define*

$$D = \{(x, y) \in X \times X \mid x \sim y\}.$$

*Then  $X/\sim$  is Hausdorff if and only if  $D$  is closed in  $X \times X$ .*

*Proof.* Let us first show that  $D$  is closed if  $X/\sim$  is Hausdorff. We have that

$$D = (p \times p)^{-1}(\Delta),$$

where  $\Delta = \{(x, x) \in (X/\sim) \times (X/\sim)\}$  is the diagonal. Since  $p \times p$  is continuous and  $\Delta$  is closed by the assumption that  $X/\sim$  is Hausdorff,  $D$  is also closed. We did not need  $p$  to be open for this direction. Conversely, if  $(X \times X) - D$  is open in  $X \times X$  then  $(p \times p)((X \times X) - D)$  is open in  $(X/\sim \times X/\sim)$  because  $p$  is assumed to be an open map. But that image is  $(X/\sim \times X/\sim) - \Delta$  because  $p$  is surjective.  $\square$

## 1.4 Compactness

A collection  $\mathcal{U}$  of open subsets of a space  $X$  is called an *open cover* if their union is all of  $X$ . The following definition is of central importance in topology:

**Definition 1.4.1.** *A space  $X$  is compact if any open cover of  $X$  contains a finite subcover, i. e. we can choose a finite subset  $\mathcal{V} \subseteq \mathcal{U}$  which is still a cover.*

To check compactness using the definition can be awkward. The following is a useful criterion for compactness:

**Theorem 1.4.2** (Heine-Borel). *Any closed and bounded subset of  $\mathbb{R}^n$  is compact.*

We begin with proving a lemma:

**Lemma 1.4.3.** *Closed subsets of compact sets are compact.*

*Proof.* Let  $A$  be a closed subset of a compact set  $X$ , and let  $\mathcal{U}$  be an open cover of  $A$ . Then  $\mathcal{U}' = \mathcal{U} \cup \{X - A\}$  is an open cover of  $X$  since  $X - A$  is open. By the compactness of  $X$ , it must contain a finite subcover  $\mathcal{V}' = \mathcal{V} \cup \{X - A\}$ . (The set  $X - A$  must be part of it unless  $X = A$ , in which case there is nothing to prove.) Then  $\mathcal{V}$  is a finite subcover of  $\mathcal{U}$  of  $A$ .  $\square$

*Proof of the Heine-Borel Theorem.* By Lemma 1.4.3, it suffices to show that for any  $a > 0$ , the cube  $Q_0 = [-\frac{a}{2}, \frac{a}{2}]^n \subseteq \mathbb{R}^n$  is compact because by definition, a bounded set must lie in such a cube.

We will prove this by contradiction. Let  $\mathcal{U}$  be an open cover of  $Q_0$  that does not have a finite subcover. Divide  $Q_0$  into  $2^n$  subcubes of half the side length; at least one of these cubes, let's say  $Q_1$ , cannot be covered by finitely many elements from  $\mathcal{U}$ . Continue in this way, producing nested cubes  $Q_i$  with side lengths  $\frac{a}{2^i}$ .

Now choose a point  $x_i \in Q_i$  for each  $i$ . This sequence is Cauchy and thus, because of the completeness of  $\mathbb{R}^n$ , has a limit  $x$ . By Lemma 1.1.4,  $x \in Q_i$  for all  $i$ . Now let  $U \in \mathcal{U}$  be a set containing  $x$ . Then  $B_\epsilon(x) \subseteq U$  for some  $\epsilon > 0$  because  $U$  is open. But then for  $i$  large enough,  $Q_i \subseteq B_\epsilon(x) \subseteq U$ , showing that  $Q_i$  did not, after all, need infinitely many members of  $\mathcal{U}$  to be covered, but only one.  $\square$

*Example 1.4.4.* The unit sphere  $S^n \subseteq \mathbb{R}^{n+1}$ , consisting of all vectors of length 1, is compact. Indeed, it is defined as the preimage of the closed set  $\{1\} \subseteq \mathbb{R}$  under the continuous map

$$| - |: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{\geq 0}$$

and hence is closed; that it is bounded is part of the definition.

**Lemma 1.4.5.** *Let  $X$  be a compact topological space and  $\sim$  an equivalence relation on  $X$ . Then  $X/\sim$  is also compact.*

*Proof.* As before, let  $p: X \rightarrow X/\sim$  denote the quotient map. Let  $\mathcal{V} = \{V_i\}_{i \in I}$  be an open cover of  $X/\sim$ . Then  $\{p^{-1}(V_i)\}_{i \in I}$  is an open cover of  $X$  by the definition of the quotient topology. Hence it contains a finite subcover  $\{p^{-1}V_{i_1}, \dots, p^{-1}V_{i_n}\}$ . But then  $\{V_{i_1}, \dots, V_{i_n}\}$  is a cover of  $X/\sim$  as well.  $\square$

*Remark 1.4.6.* In algebraic geometry, the term “compact” is often understood to mean “compact and Hausdorff”; what we call compact here would be called “quasicompact”.

## 1.5 Countability

This section is somewhat technical but required for the correct definition of manifolds later on.

**Definition 1.5.1.** *A space  $X$  with topology  $\mathcal{U}$  is said to satisfy the second countability axiom, or shorter, to be second-countable, if there exists a countable subset  $\mathcal{V}$  of  $\mathcal{U}$  which is closed under finite intersections and such that  $\mathcal{U}$  is the smallest topology containing  $\mathcal{V}$ .*

Second-countability is thus a condition that restricts the size of the topology on a space. For instance, an uncountable set with the discrete topology is not second-countable. An equivalent way of phrasing the condition is that every open set  $U \in \mathcal{U}$  can be written as the union of those open sets in  $\mathcal{V}$  that lie in  $U$ :

$$U = \bigcup_{\substack{V \in \mathcal{V} \\ V \subseteq U}} V. \quad (1.1)$$

Since there are at most  $2^{|\mathcal{V}|}$  different ways of taking arbitrary unions of elements of  $\mathcal{V}$ , a second-countable topology can not have more open sets than the cardinality of  $\mathbb{R}$ .

The reader may be curious as to what the first countability axiom says. We will not be needing it here, but for the sake of completeness, it is local second-countability: a space is first-countable if every point  $x$  is contained in some open set  $U$  that is second-countable.

**Lemma 1.5.2.** *For any  $n$ , the standard topology on  $\mathbb{R}^n$  is second-countable.*

*Proof.* Let  $\mathcal{V} = \{B_q(x) \mid q \in \mathbb{Q}, x \in \mathbb{Q}^n\}$ . Then  $\mathcal{V}$  is countable, and (1.1) holds basically by the completeness of  $\mathbb{R}^n$ .  $\square$

**Lemma 1.5.3.** *Any subspace of a second-countable space is second-countable. A quotient space  $X/\sim$  of a second-countable space  $X$  is second-countable if the projection  $p: X \rightarrow X/\sim$  is open.*

*Proof.* Let  $X$  be second countable with a countable subset  $\mathcal{V}$  satisfying (1.1). If  $A \subseteq X$  then  $\mathcal{V}' = \{V \cap A \mid V \in \mathcal{V}\}$  is a countable subset of the topology of  $A$  satisfying (1.1) for open subsets  $U$  of  $A$ .

For the statement about quotients, the open sets of  $X/\sim$  are exactly the images under the open projection map of opens in  $X$ , hence  $\mathcal{V}' = \{p(V) \mid V \in \mathcal{V}\}$  fits the bill.  $\square$

## Chapter 2

# The definition of projective space

The  $n$ -dimensional real projective space is defined to be the set of all lines through the origin in  $\mathbb{R}^{n+1}$ ; a similar definition works for the complex projective space or, in fact, projective spaces over any ring  $k$ . It is denoted by  $\mathbb{R}P^n$  resp.  $\mathbb{C}P^n$  by topologists and  $P^n(\mathbb{R})$  resp.  $P^n(\mathbb{C})$  (or  $\mathbb{P}^n(\mathbb{R})$  or  $\mathbb{P}_{\mathbb{R}}^n$  etc.) by algebraic geometers:

$$\mathbb{R}P^n = P^n(\mathbb{R}) = \{L \leq \mathbb{R}^{n+1} \mid L \text{ 1-dimensional linear subspace}\}.$$

Unfortunately, using this definition it is quite awkward to try to define a sensible topology on  $\mathbb{R}P^n$ ; we will do this in the next section.

### 2.1 Projective spaces as quotients

A line through the origin in  $\mathbb{R}^{n+1}$  is uniquely determined by any other point on that line, that is, any point  $x \in \mathbb{R}^{n+1} - \{0\}$ . That is, we have a surjective map

$$p: \mathbb{R}^{n+1} - \{0\} \rightarrow \mathbb{R}P^n$$

sending a point  $(x_0, \dots, x_n)$  to the subspace spanned by that vector. This map is surjective, but clearly not injective since a line contains many points. More precisely, we define an equivalence relation  $\sim$  on  $\mathbb{R}^{n+1} - \{0\}$  by decreeing that two points  $x, y$  are equivalent if and only if they both lie on a line through the origin, that is,  $(x_0, \dots, x_n) \sim (y_0, \dots, y_n)$  if there is a  $\lambda \in \mathbb{R} - \{0\}$  such that  $x_i = \lambda y_i$  for all  $0 \leq i \leq n$ .

It is now clear that  $p$  factors through a bijective map

$$\bar{p}: (\mathbb{R}^{n+1} - \{0\}) / \sim \rightarrow \mathbb{R}P^n.$$

We define the standard topology on  $\mathbb{R}P^n$  to be the quotient topology under this identification.

Another way of looking at this construction is to define a group action of the multiplicative group  $\mathbb{R}^\times$  on  $\mathbb{R}^{n+1} - \{0\}$  by scalar multiplication. Then  $\mathbb{R}P^n \cong (\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^\times$ .

The equivalence class of a point  $(x_0, \dots, x_n)$  in  $\mathbb{R}P^n$  is customarily denoted by  $[x_0 : \dots : x_n]$  or  $(x_0 : \dots : x_n)$ .

We will give another couple of constructions of  $\mathbb{R}P^n$  and  $\mathbb{C}P^n$  as quotients. These alternative construction will be useful later.

**Lemma 2.1.1.** *Let  $S^{2n+1} = \{z \in \mathbb{C}^{n+1} \mid |z| = 1\} \subseteq \mathbb{C}^{n+1} - \{0\}$  denote the  $2n + 1$ -dimensional standard sphere. The group  $S^1$  of complex numbers of absolute value 1 acts on  $S^{2n+1}$  by scalar multiplication. Then*

$$\mathbb{C}P^n \cong S^{2n+1}/S^1.$$

Similarly,  $\mathbb{R}P^n \cong S^n/\{\pm 1\}$ .

*Proof.* We will only prove the complex case. The inclusion  $S^{2n+1} \hookrightarrow \mathbb{C}^{n+1} - \{0\}$  is equivariant with respect to the  $S^1$ -action, thus we get an induced continuous map

$$S^{2n+1}/S^1 \rightarrow (\mathbb{C}^{n+1} - \{0\})/S^1 \twoheadrightarrow (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^\times,$$

where the last map is the projection associated to the group inclusion  $S^1 \subset \mathbb{C}^\times$ . It is clear that this map is bijective. The inverse map can be described as

$$(\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^\times \rightarrow S^{2n+1}/S^1; \quad [x_0 : \dots : x_n] \mapsto \left[ \frac{1}{|x|} (x_0, \dots, x_n) \right],$$

which is continuous, thus the desired homeomorphism is established.  $\square$

If we write  $S^n$  as a union of its upper and its lower hemisphere,  $S^n = D_+^n \cup D_-^n$  (both parts including the equator), we observe that for every point  $x \in S^n$ , either  $x \in D_+^n$  or  $-x \in D_+^n$ . Thus we have proved:

**Lemma 2.1.2.** *There is a homeomorphism  $\mathbb{R}P^n \cong D^n/\sim$ , where  $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$  and the equivalence relation  $\sim$  identifies antipodal points on the boundary  $S^{n-1}$ .*

A similar construction works for  $\mathbb{C}P^n$  and is left to the reader.

## 2.2 $\mathbb{R}P^1$ and $\mathbb{C}P^1$

**Lemma 2.2.1.** *There are homeomorphisms  $\mathbb{R}P^1 \cong S^1$  and  $\mathbb{C}P^1 \cong S^2$ .*

*Proof.* Consider the map  $f: S^1 \rightarrow S^1$  given by  $f(z) = z^2$ , where we think of  $z \in S^1 \subseteq \mathbb{C}$  as a unit complex number. Then  $f$  is surjective and  $f(z) = f(-z)$ , thus it factors through a bijection

$$\bar{f}: S^1/\{\pm 1\} \cong \mathbb{R}P^1 \rightarrow S^1.$$

Moreover,  $\bar{f}$  is a homeomorphism because it is continuous and open, the latter because  $f$  is open.

For the complex case, we have to employ some other methods. We think of the 2-sphere  $S^2$  as the “one-point compactification” of  $\mathbb{C} = \mathbb{R}^2$ . The stereographic projection gives a homeomorphism  $\sigma: S^2 - \{N\} \rightarrow \mathbb{C}$ , where  $N$  denotes the north pole of  $S^2$ . Now we define

$$f: \mathbb{C}P^1 \rightarrow S^2$$

by

$$f([z_1 : z_2]) = \begin{cases} \sigma^{-1}\left(\frac{z_1}{z_2}\right); & \text{if } z_2 \neq 0 \\ N; & \text{if } z_2 = 0. \end{cases}$$

This is well-defined and bijective; we have to check that it is continuous and open. That it is that away from the point  $[1 : 0]$  resp. the north pole is obvious.  $\square$

## 2.3 $\mathbb{R}P^2$

The projective plane  $\mathbb{R}P^2$  is an example of a *non-orientable surface*; it can be obtained by taking a Möbius strip and attaching a two-dimensional disk along its single boundary circle. This cannot be embedded in  $\mathbb{R}^3$ , but it can be embedded into  $\mathbb{R}^4$ :

**Lemma 2.3.1.** *The map  $f: S^2 \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^4$  given by*

$$f(x, y, z) = (xy, xz, y^2 - z^2, 2yz)$$

*induces an embedding of  $\mathbb{R}P^2$ .*

*Proof.* First note that if we change signs on  $x, y, z$  simultaneously, the image of  $f$  does not change, thus  $f$  is well-defined on  $\mathbb{R}P^2$ . We leave it to the reader to check that this map is injective and closed.  $\square$

## 2.4 $\mathbb{R}P^3$

**Lemma 2.4.1.** *There is a homeomorphism  $SO(3) \cong \mathbb{R}P^3$ .*

*Proof.* Recall from Lemma 2.1.2 that we can describe  $\mathbb{R}P^3$  as a quotient of  $D^3$ , where antipodal points on the boundary are identified. Define a map

$$\phi: D^3 \rightarrow SO(3)$$

as follows: a point  $\alpha x \in D^3$ , where  $0 \leq \alpha \leq 1$  and  $x \in \mathbb{R}^3$  is a unit vector, is mapped to the rotation around  $x$  by  $\alpha\pi$  in the positive direction. This factors through  $\mathbb{R}P^3$  because a rotation by  $\pi$  and the rotation by  $\pi$  in the other direction are the same. One can write down a more explicit formula for this map and verify that it is continuous; this is left to the reader. We will, however, show that the map is bijective. Injectivity is obvious since all the rotations thus produced are distinct. To see that every element in  $SO(3)$  is a rotation, and hence in the image of  $\phi$ , let  $\lambda_1, \lambda_2, \lambda_3$  be the three complex eigenvalues of  $A \in SO(3)$ . Then one of the  $\lambda_i$  has to be real (a degree-3 real polynomial has a real root), while the other two are complex conjugated, and  $\lambda_1\lambda_2\lambda_3 = \det A = 1$ . This can only happen if 1 is one of the eigenvalues. If  $x$  is an associated eigenvector, it spans a stable axis, and  $A$  is some rotation around this axis.  $\square$

# Chapter 3

## Topological properties of projective spaces

### 3.1 Point-set topological properties

**Proposition 3.1.1.** *The projective spaces  $\mathbb{R}P^n$  and  $\mathbb{C}P^n$  are compact.*

*Proof.* By Lemma 2.1.1,  $\mathbb{R}P^n$  and  $\mathbb{C}P^n$  are quotients of spheres. Spheres are compact by Theorem 1.4.2, and quotients of compact spaces are compact by Lemma 1.4.5.  $\square$

**Proposition 3.1.2.** *The projective spaces  $\mathbb{R}P^n$  and  $\mathbb{C}P^n$  are Hausdorff.*

*Proof.* We will give the proof for  $\mathbb{C}P^n$  only. Define a map

$$f: (\mathbb{C}^{n+1} - \{0\}) \times (\mathbb{C}^{n+1} - \{0\}) \rightarrow \mathbb{R}$$

by

$$f(x, y) = f(x_1, \dots, x_n, y_1, \dots, y_n) = \sum_{i \neq j} |x_i y_j - x_j y_i|^2.$$

We observe that  $f(x, \lambda x) = 0$  for all  $\lambda \in \mathbb{C}$ . Conversely, if  $f(x, y) = 0$  then  $x_i y_j = x_j y_i$  for all  $i \neq j$  and thus  $x$  and  $y$  are linearly dependent.

We thus conclude that  $f^{-1}(0) = \{(x, y) \mid x \sim y\}$  under the equivalence relation of linear dependence that defines  $\mathbb{C}P^n$  as a quotient. Since  $\{0\} \subseteq \mathbb{R}$  is closed and  $f$  is continuous,  $\{(x, y) \mid x \sim y\}$  is closed in  $(\mathbb{C}^{n+1} - \{0\}) \times (\mathbb{C}^{n+1} - \{0\})$ . By Lemma 1.3.3,  $\mathbb{C}P^n$  is Hausdorff.  $\square$

**Proposition 3.1.3.** *The projective spaces  $\mathbb{R}P^n$  and  $\mathbb{C}P^n$  are second countable.*

*Proof.* They are open quotients of subspaces of  $\mathbb{R}^n$  and hence second countable by Lemmas 1.5.2, 1.5.3, and 1.2.2.  $\square$

## 3.2 Charts and manifold structures

An important property of projective spaces is that they are smooth manifolds.

**Definition 3.2.1.** *Let  $M$  be a second countable Hausdorff space. A chart on  $M$  is a homeomorphism  $\phi: U \rightarrow V$ , where  $U$  is an open subset of  $M$  and  $V$  is an open subset of  $\mathbb{R}^n$  for some  $n$ . An atlas on  $M$  is a collection of charts  $\phi_\alpha: U_\alpha \rightarrow V_\alpha$  such that the  $U_\alpha$  together cover  $M$ .*

*The space  $M$  is called a topological manifold if it has an atlas.*

This is a useful definition, but not quite what we're after; we want to do analysis on manifolds, so there should be a notion of "differentiable function" on it.

**Definition 3.2.2.** *An atlas  $(\phi_\alpha: U_\alpha \rightarrow V_\alpha)_\alpha$  on a topological manifold  $M$  is called smooth if whenever  $U_\alpha$  and  $U_\beta$  have nontrivial intersection  $U_{\alpha\beta}$ , the map*

$$\phi_\alpha(U_{\alpha\beta}) \xrightarrow{\phi_\alpha^{-1}} U_{\alpha\beta} \xrightarrow{\phi_\beta} \phi_\beta(U_{\alpha\beta})$$

*is not only a homeomorphism but a diffeomorphism of open subsets of  $\mathbb{R}^n$ .*

*A function  $f: M \rightarrow \mathbb{R}$  is smooth with respect to a smooth atlas  $\phi_\alpha$  if  $f \circ \phi_\alpha^{-1}: V_\alpha \rightarrow \mathbb{R}$  is smooth. A function  $f: M \rightarrow N$  between manifolds with smooth atlases is called smooth if for every smooth function  $g: N \rightarrow \mathbb{R}$ , the function  $g \circ f: M \rightarrow \mathbb{R}$  is also smooth.*

We would now like to say that a "smooth manifold" is a topological manifold together with a smooth atlas. Although one can read such a statement in the literature, this is not correct. Two different smooth atlases can give rise to the same class of smooth functions and in that case, we do not want to consider those manifolds as different.

**Definition 3.2.3.** *A smooth manifold  $M$  is a topological manifold together with an equivalence class of smooth atlases. Here an atlas  $\phi_\alpha$  is equivalent to an atlas  $\psi_\beta$  if the identity map  $\text{id}: (M, \phi_\alpha) \rightarrow (M, \psi_\beta)$  is smooth.*

We will sometimes omit the word "smooth" and just speak of a "manifold", it being understood that it is smooth.

**Theorem 3.2.4.** *The projective spaces  $\mathbb{R}P^n$  and  $\mathbb{C}P^n$  are smooth manifolds of dimensions  $n$  and  $2n$ , respectively.*

*Proof.* We have already seen that projective spaces are second countable and Hausdorff. Let  $k = \mathbb{R}$  or  $\mathbb{C}$ . We define charts on  $kP^n$  as follows:

$$U_i = \{[x_0 : \cdots : x_n] \in kP^n \mid x_i \neq 0\} \quad (i = 0, \dots, n).$$

Then we have homeomorphisms

$$\phi_i: U_i \rightarrow k^n; \quad [x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}\right),$$

where the  $i$ th entry (which would be  $\frac{x_i}{x_i} = 1$ ) is omitted. Clearly the  $U_i$  cover  $kP^n$  as every point in  $kP^n$  has some nonzero coordinate. Moreover, the map

$$\psi_i: k^n \rightarrow U_i; \quad (x_1, \dots, x_n) \mapsto [x_1 : \cdots : 1 : \cdots : x_n],$$

where the entry 1 is in the  $i$ th slot, is a continuous inverse for  $\phi_i$ .

Now consider the change-of-coordinate functions  $\phi_i \circ \phi_j^{-1}$  (for ease of notation, let's assume  $i < j$ ):

$$(x_1, \dots, x_n) \mapsto [x_1, \dots, 1, \dots, x_n] \mapsto \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{j-1}}{x_i}, \frac{1}{x_i}, \frac{x_{j+1}}{x_i}, \dots, \frac{x_n}{x_i}\right).$$

This is clearly a smooth function, thus the  $\phi_i$  exhibit a smooth atlas for  $kP^n$ .  $\square$

### 3.3 Cell structures

A manifold structure on a space of interest, like projective spaces, is crucial for its geometric study, but for its topological properties, it is often more useful to have a more combinatorial description. Topologists like to work with the category of so-called *CW complexes*. To get an intuition for this, consider what a graph is: it consists of vertices (0-dimensional “cells”) and edges (1-dimensional “cells”), and the end points of the edges are identified (glued) to certain vertices. A CW-complex is a higher-dimensional generalization of this.

Denote by  $D^n$  the standard  $n$ -dimensional disk of vectors in  $\mathbb{R}^n$  of norm  $\leq 1$ ; its boundary is the sphere  $S^{n-1}$ .

**Definition 3.3.1.** Let  $\phi_\alpha: S^{n-1} \rightarrow X$  be a collection of maps. Then we define a new space

$$Y = X \cup_{\phi_\alpha} \coprod_{\alpha} D^n = (X \sqcup \coprod_{\alpha} D^n) / \sim,$$

where the equivalence relation is given by  $\phi_\alpha(x) \sim x_\alpha$ . Here  $x \in S^{n-1}$  and  $x_\alpha$  denotes the point  $x$  in the  $\alpha$ th summand of  $\coprod_{\alpha} D^n$ .

We call  $\phi_\alpha$  the attaching maps, the various  $D^n$   $n$ -cells, and say that  $Y$  is obtained from  $X$  by attaching (a number of)  $n$ -cells.

**Definition 3.3.2.** A CW-complex is a topological space  $X$  with a filtration

$$X^{(0)} \subseteq X^{(1)} \subseteq \dots \subseteq X$$

such that:

- $X^{(0)}$  is discrete,
- $X^{(n)}$  is obtained from  $X^{(n-1)}$  by attaching  $n$ -cells for all  $n > 0$ .
- $\bigcup_{n \geq 0} X^{(n)} = X$ , and

The subspace  $X^{(n)}$  is called the  $n$ -skeleton. A CW-complex is said to be of dimension  $n$  if  $X^{(n)} = X^{(n+1)} = \dots = X$ . A CW-complex is finite if it is of finite dimension and  $X^n$  is obtained from  $X^{n-1}$  by attaching only finitely many cells.

**Theorem 3.3.3.** The projective space  $\mathbb{R}P^n$  is obtained from  $\mathbb{R}P^{n-1}$  by attaching a single  $n$ -cell. Also  $\mathbb{C}P^n$  is obtained from  $\mathbb{C}P^{n-1}$  by attaching a single  $2n$ -cell.

In particular,  $\mathbb{R}P^n$  and  $\mathbb{C}P^n$  are finite,  $n$ - resp.  $2n$ -dimensional CW-complexes with exactly one cell in every resp. every even dimension.

*Proof.* By Lemma 2.1.2,

$$\mathbb{R}P^n \cong D^n / \sim,$$

where the equivalence relation  $\sim$  identifies antipodal points on the boundary  $S^{n-1}$ . Define  $\phi: S^{n-1} \rightarrow \mathbb{R}P^{n-1}$  to be the standard quotient map. Then  $\mathbb{R}P^n \cong \mathbb{R}P^{n-1} \cup_{\phi} D^n$ .

A similar construction works for  $\mathbb{C}P^n$  and is left to the reader. □

### 3.4 Euler characteristic

A classical result from graph theory is that for every finite planar graph  $\Gamma = (V, E)$ , i. e. a graph one can embed onto a 2-dimensional sphere, the number  $\chi(\Gamma) = \#V - \#E + \#F$ , the difference between the number of vertices and the number of edges plus the number of 2-dimensional faces, is always 2. This says that this number 2, called the Euler number, is an invariant of the sphere itself and independent of the graph. The number will be different if we allow ourselves to embed the graph e.g. on a donut (what will it be then?). We can think of a graph embedded in a surface as giving rise to a 2-dimensional CW-complex with 1-skeleton the graph itself and 2-cells the faces of the graph.

The following theorem requires some knowledge of algebraic topology and is beyond our scope in this course:

**Theorem 3.4.1.** *Let  $X$  be a finite CW-complex, and let  $n_i$  denote the number of  $i$ -cells. Then the Euler characteristic*

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i n_i$$

*is independent of the CW-structure.*

*Example 3.4.2.* A point has Euler characteristic 1 because it consists of a single 0-cell. The  $n$ -dimensional sphere has Euler characteristic 2 for even  $n$  and 0 for odd  $n$  because it can be given a CW-structure with one 0-cell and one  $n$ -cell.

*Example 3.4.3.* The torus  $S^1 \times S^1$  has Euler characteristic 0 because it can be given a CW structure with one 0-cell, two 1-cells (the longitudinal and latitudinal great circles), and one 2-cell.

**Proposition 3.4.4.** *We have that  $\chi(\mathbb{C}P^n) = n+1$  and  $\chi(\mathbb{R}P^n) = \begin{cases} 0; & n \text{ odd} \\ 1; & n \text{ even} \end{cases}$*

*Proof.* This follows directly from the CW structure of  $\mathbb{R}P^n$  and  $\mathbb{C}P^n$  from Thm. 3.3.3. □