Algebra and geometry through projective spaces

Problems on curves in the projective plane

due: Monday, March 9

Problem 1. Let C be a conic passing throught the point $[0 : 0 : 1]$, i.e., having equation of the form

$$ax^2 + bxy + cy^2 + dxz + eyz = 0.$$

Show that C has is reducible if and only if $cd^2 - bde + ae^2 = 0$ under the assumption that C has at least two rational points.

Problem 2. Find the equation of the hypersurface definied by the image of the map $\Phi\colon \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^5$ defined by

$$([s_1 : t_1 : u_1], [s_2 : t_2 : u_2]) \mapsto (s_1s_2 : s_1t_2 + t_1s_2 : t_1t_2 : s_1u_2 + u_1s_2 : t_1u_2 + u_1t_2 : u_1u_2].$$

Problem 3. Find the normal form for the Fermat cubic $x^3 + y^3 = z^3$.

Problem 4. Show that an elliptic curve over $\mathbb{R}$ cannot have more than three flex points.

Problem 5. Define the rational cubic curve C as the image of the map $\Phi\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ given by

$$\Phi([s : t]) = [s^3 : st^2 : t^3], \qquad [s : t] \in \mathbb{P}^1.$$

Find the singular point of C and determine whether C is nodal or cuspidal.

Problem 6. Let C be a cubic plane curve over $\mathbb{C}$. Show that the Hessian, i.e., the determinant of

$$\begin{bmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial x \partial z} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y \partial z} \\ \frac{\partial^2 F}{\partial z \partial x} & \frac{\partial^2 F}{\partial z \partial y} & \frac{\partial^2 F}{\partial z^2} \end{bmatrix}$$

vanishes exactly at the singular points of C and on the flex points of C.

Problem 7. Show that if P is a non-singular point of C_1 and C_2 such that the tangents of C_1 and C_2 at P are distinct, then $I_P(f, g) = 1$ where f and g are the homogeneous polynomials defining C_1 and C_2.

Problem 8. Follow the proof of Bézout's Theorem given in the notes starting with the curves $zy^2 = x^3 - xz^2$ and $x^2 + y^2 = z^2$. What are all the intersection points and their multiplicities at the end of the reduction?