Exercises. Deadline is 1st of April

There following six exercises are to be solved, neatly explained, and handed in as a .pdf file, before the end of April 1.

1.

Show that $\mathbb{A}^n$ is compact; for any open cover $\{U_\alpha\}$ of $\mathbb{A}^n$ a finite subcollection will be a covering.

2.

Let $Z \subseteq \mathbb{A}^3$ be the algebraic set defined by the ideal

$$I = (x^2 - yz, xz - x) \subset \mathbb{C}[x, y, z].$$

Show that Z has three components, and describe their corresponding prime ideals

3.

Consider the quadratic surface $Q \subset \mathbb{P}^3$ given by the polynomial $XY - ZW$ in $\mathbb{C}[X, Y, Z, W]$. Show that Q contains two families of lines L_P and N_P parametrized by points $P = [t : u] \in \mathbb{P}^1$, with the following property.

$$L_P \cap L_{P'} = N_P \cap N_{P'} = \emptyset \quad \text{if} \quad P \neq P',$$

and

$$L_P \cap N_{P'} = \text{a point} \quad \text{for all } P, P'.$$

Hint, use that $XY - ZW$ is the determinant of the matrix

$$\begin{bmatrix} X & Z \\ W & Y \end{bmatrix}.$$

4.

Let $Y \subseteq \mathbb{A}^3$ be the twisted cubic, that is the image of $\mathbb{A}^1 \longrightarrow \mathbb{A}^3$ by the map $t \mapsto (t, t^2, t^3)$. Use this example to show that if $f_1, \ldots, f_r$ generates an ideal $I(Y)$, then $h(f_1), \ldots, h(f_r)$ does not generate the ideal $I(\overline{Y})$ of its closure in projective space. Here $h(f)$ is the homogenization of a polynomial, defined in Section 9.1.

5.

Let $V \subseteq \mathbb{P}^5$ denote the Veronese surface. Show that for any two points P and Q on V there exists a conic curve C on V passing through the points P and Q. The curve C will be an embedding of the projective line $\mathbb{P}^1$.

6.

Prove "The image of the Segre map $\sigma\colon \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}$ is the Segre varietey $\Sigma_{m,n}$ defined by the homogeneous ideal

$$I = (Z_{i,j}Z_{k,l} - Z_{i,l}Z_{k,j}) \subset \mathbb{C}[Z_{i,j}]_{\substack{0\leq i\leq m,\\ 0\leq j\leq n}}$$

for all i, j, k, l. "