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Contents

Chapter 7

Affine Varieties

7.1 The polynomial ring

Let $\mathbb{C}$ denote the field of complex numbers, and let $\mathbb{C}[x_1, \ldots, x_n]$ denote the polynomial ring in n variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{C}$. Elements f in $\mathbb{C}[x_1, \ldots, x_n]$ are polynomials in $x_1, \ldots, x_n$, that is finite expressions of the form

$$f = \sum c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

with c_α in $\mathbb{C}$. Polynomials are added and multiplied in the obvious way, and $\mathbb{C}[x_1, \ldots, x_n]$ indeed forms a ring; a commutative unital ring.

7.2 Hypersurfaces

To any $f \in \mathbb{C}[x_1, \ldots, x_n]$ we let $Z(f)$ denote the zero set of the element f, that is

$$Z(f) = \{(a_1, \ldots, a_n) \in \mathbb{C}^n \mid f(a_1, \ldots, a_n) = 0\}.$$

For non-constant polynomials f the zero set $Z(f)$ is referred to as a hypersurface. Clearly we have that the union satisfies

$$Z(f) \cup Z(g) = Z(fg).$$

In order to describe intersections of hypersurfaces it is convenient to use ideals, a notion we recall next.

7.3 Ideals

A non-empty subset $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ that is closed under sum, and closed under multiplication by elements of $\mathbb{C}[x_1, \ldots, x_n]$, is called an ideal. The zero element is an ideal, and the whole ring is an ideal.

If $\{f_\alpha\}_{\alpha \in \mathscr{A}}$ is a collection of elements in $\mathbb{C}[x_1, \ldots, x_n]$ they generate the ideal $I(f_\alpha)_{\alpha \in \mathscr{A}}$ that consists of all finite expressions of the form

$$I(f_\alpha)_{\alpha \in \mathscr{A}} = \{\sum_{\alpha \in \mathscr{A}} g_\alpha f_\alpha \mid g_\alpha \in \mathbb{C}[x_1, \ldots, x_n], g_\alpha \neq 0 \text{ finite indices } \alpha\}.$$

The zero ideal is generated by the element 0, and this is the only element that generates the zero ideal. We have that $(1) = \mathbb{C}[x_1, \ldots, x_n]$, so the element 1 generates the whole ring.

Noetherian ring

The polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ is an example of a Noetherian ring which means that any ideal is in fact finitely generated. Thus, if I is an ideal generated by the collection $\{f_\alpha\}_{\alpha \in \mathscr{A}}$, then there exists a finite subset $f_1, \ldots, f_m$ of the collection that generates the ideal $I(f_\alpha)_{\alpha \in \mathscr{A}} = I(f_1, \ldots, f_m)$.

7.4 Algebraic sets

Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be an ideal. We let $Z(I)$ denote the intersection of the zero sets of the elements in I, that is

$$Z(I) = \cap_{f \in I} Z(f).$$

A subset of $\mathbb{C}^n$ of the form $Z(I)$ for some ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is called an *algebraic set*. One verifies that if the ideal I is generated by $f_1, \ldots, f_m$ then we have that

$$Z(I) = Z(f_1) \cap Z(f_2) \cap \cdots \cap Z(f_m).$$

In particular we have that $Z(f) = Z(I(f))$, where $I(f)$ is the ideal generated by f.

Union and intersection

Let $\{I_\alpha\}_{\alpha \in \mathscr{A}}$ be a collection of ideals in $\mathbb{C}[x_1, \ldots, x_n]$. Their set theoretic intersection is an ideal we denote by $\cap_{\alpha \in \mathscr{A}} I_\alpha$. Their union $I(\cup_\alpha I_\alpha)$ is the ideal generated by their set theoretic union. If the collection is finite, we have the product $I_1 \cdots I_m$, which denotes the ideal where the elements are finite sums of products $f_1 \cdots f_m$, with $f_i \in I_i$, for $i = 1, \ldots, m$.

7.5 Zariski topology

Lemma 7.5.1. *We have that the algebraic sets in $\mathbb{C}^n$ satisfy the following properties.*

(1) Finite unions $\cup_{i=1}^m Z(I_i) = Z(I_1 \cdots I_m) = Z(\cap_{i=1}^m I_i)$.

(2) Arbitrary intersections $\cap_\alpha Z(I_\alpha) = Z(I(\cup_\alpha I_\alpha))$.

We have furthermore that $Z(1) = \emptyset$ and that $Z(0) = \mathbb{C}^n$.

Proof. This is an excellent exercise. □

The lemma above shows that the collection of algebraic sets satisfy the axioms for the *closed* sets of a topology. This particular topology where the closed sets are the algebraic sets is called the Zariski topology.

Open sets

When defining the topology on a set, it is customary to define what the open sets are. The open sets are the complements of the closed sets, so having defined what the closed sets are we also know what the open sets are. But, we could have started the other way around. A collection $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ of subsets of a set X that contains X and $\emptyset$, and which is closed under finite intersections, and arbitrary unions, define the open sets of a topology on X.

7.6 Affine varieties

Definition 7.6.1. *We let $\mathbb{A}^n$ denote the vector space $\mathbb{C}^n$ endowed with the Zariski topology. The space $\mathbb{A}^n$ is called the affine n-space.*

Example 7.6.2. The affine plane is our favourite example. For any element $f \in \mathbb{C}[x, y]$ the zero zet $Z(f)$ is by defintion a closed set. The intersection of two hypersurfaces - or curves - is

$$Z(f) \cap Z(g) = Z(f, g),$$

the collection of points corresponding to their common intersections - which typically is a finite set of points. For instance, let $f = x - y + 1$ and $g = y^2 - x^3$.

Example 7.6.3. Show that the open sets $D(f) = \mathbb{A}^n \setminus Z(f)$, with $f \in \mathbb{C}[x_1, \ldots, x_n]$ form a basis for the topology. That is, any open can be written as a union of the basic opens $D(f)$. In the usual, strong, topology, the open balls form a basis for the topology.

Example 7.6.4. The open sets in $\mathbb{A}^n$ are big: Show that any two non-empty opens U and V in $\mathbb{A}^n$ has a non-empty intersection. In particular we get that $\mathbb{A}^n$ is not a Hausdorff space.

Example 7.6.5. Show that $\mathbb{A}^n$ is compact; for any open cover $\{U_\alpha\}$ of $\mathbb{A}^n$ a finite subcollection will be a covering.

Definition 7.6.6. *A (non-empty) topological space X is called irreducible if X can not be written as the union $X = X_1 \cup X_2$ of two proper closed subsets X_1 and X_2 of X.*

Example 7.6.7. The affine line $\mathbb{A}^1$ is irreducible. Because any non-zero polynomial f is such that $Z(f)$ is a collection of finite points. It follows that closed, proper, subsets of $\mathbb{A}^1$ are collections of finite points. And in particular we can not write $\mathbb{A}^1$ as a union of two finite sets, hence $\mathbb{A}^1$ is irreducible.

Example 7.6.8. If X is irreducible, then any non-empty open $U \subseteq X$ is also irreducible (Excercise 7.6.13). In particular if we let $U = \mathbb{A}^1 \setminus (0)$, then U is irreducible even if the picture you draw indicates that the space U is not even connected. A topological space X is not connected if it can be written as a union $X = X_1 \cup X_2$ of two proper closed subsets where $X_1 \cap X_2 = \emptyset$. In particular a space that is not connected is in particular not irreducible.

Definition 7.6.9. *An irreducible algebraic set is an affine algebraic variety.*

Example 7.6.10. Let $I = (xy) \subseteq \mathbb{C}[x, y]$ denote the ideal generated by the element $f = xy$. Then the algebraic set

$$Z(xy) = Z(x) \cup Z(y),$$

where the two sets $Z(x) \subset Z(xy)$ and $Z(y) \subset Z(xy)$ are proper subsets. Hence $Z(xy)$ is not an algebraic variety.

Example 7.6.11. Let $I = (x^2 + y^3) \subseteq \mathbb{C}[x, y]$. The ideal I is generated by $f = x^2 + y^3$, which is an irreducible element - which means that the element f can not be written as a product $f = f_1 \cdot f_2$ in a non-trivial way. It follows that $Z(f)$ is an algebraic variety.

Noetherian spaces

A topological space X is called Noetherian space if any descending chain of closed subsets
$$X \supseteq X_1 \supseteq \cdots X_n \supseteq X_{n+1} \cdots$$
stabilizes, that is there exists an integer n such that $X_n = X_{n+i}$, for all integers $i \geq 0$.

Excercise 7.6.12. Show that $\mathbb{A}^n$ is a Noetherian space.

Excercise 7.6.13. Let $U \subseteq X$ be a non-empty open set, with X irreducible. Show that U is also irreducible.

Excercise 7.6.14. Show that any topological space X can be written as a union of irreducible subsets, called irreducible components of the space X. If X is an algebraic variety it has only a finite set of irreducible components.

7.7 Prime ideals

Definition 7.7.1. *Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be a* proper *ideal. The ideal I is said to be a prime ideal if*
$$gf \in I \quad \textit{implies that } f \textit{ or } g \textit{ is in } I.$$

Example 7.7.2. Let $I = (xy) \subseteq \mathbb{C}[x, y]$ denote the ideal generated by the element $f = xy$. Then any element in I can be written as $F \cdot f$, with $F \in \mathbb{C}[x, y]$. In particular we get that neither x nor y is in I, but clearly xy is. Thus $I = (xy)$ is not a prime ideal.

Example 7.7.3. Let $I = (x^2 + y^3) \subseteq \mathbb{C}[x, y]$. The ideal I is generated by $f = x^2+y^3$, which is an irreducible element. It follows that the ideal $I = (x^2+y^3)$ is a prime ideal.

Excercise 7.7.4. A non-zero element $f \in \mathbb{C}[x_1, \ldots, x_n]$ is irreducible if $f = f_1 \cdot f_2$ implies that at least one of the factors f_1 or f_2 is a unit. Show that an ideal I generated by an irreducible element f implies that the algebraic set $Z(I)$ is irreducible. Give an example of an irreducibel hypersurface $Z(f)$ where f is not an irreducible element.

Excercise 7.7.5. Show that an ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is a prime ideal if and only if the quotient ring $\mathbb{C}[x_1, \ldots, x_n]/I$ is an integral domain. Recall that a ring A is called an integral domain if A is not the zero ring, and if $f \cdot g = 0$ then either $f = 0$ or $g = 0$. Show furthermore that an ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is a maximal ideal if and only if the quotient ring $\mathbb{C}[x_1, \ldots, , x_n]/I$ is a field.

Recall that an integral domain A is called a field if every non-zero element is invertible, and that a prime ideal not properly contained in any other prime ideal is maximal ideal.

Example 7.7.6. Ideals of the form $I = (x_1 - a_1, \ldots, x_n - a_n) \subseteq \mathbb{C}[x_1, \ldots, x_n]$, with $(a_1, \ldots, a_n) \in \mathbb{C}^n$, are prime ideals. For instance, the quotient ring $\mathbb{C}[x_1, \ldots, x_n]/I = \mathbb{C}$, is a field (see Exercise 7.7.5). Note that

$$Z((x_1 - a_1, \ldots, x_n - a_n)) = \cap_{i=1}^n Z(x_i - a_i) = (a_1, \ldots, a_n).$$

Excercise 7.7.7. The ideal $I = (xy) \subseteq \mathbb{C}[x, y]$ is not prime, but the ideals (x) and (y) are prime ideals. Show that $I = (x) \cap (y)$, and use this to describe the irreducible components of $Z(I)$.

7.8 Radical ideals

Definition 7.8.1. *The radical of an ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is the set*

$$\sqrt{I} = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f^m \in I \text{ some } \quad m \geq 0\}.$$

One shows that the radical $\sqrt{I}$ is also an ideal. Clearly we have an inclusion $I \subseteq \sqrt{I}$. We say that the ideal I is radical if $I = \sqrt{I}$.

Example 7.8.2. That any prime ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is a radical ideal, follows almost directly from the definitions. The converse is not the case. Consider for instance the ideal $I = (xy) \subseteq \mathbb{C}[x, y]$ discussed in Example 7.7.2. The ideal $I = (xy)$ is not prime, but radical.

Example 7.8.3. The ideal $I = (x^2) \subseteq \mathbb{C}[x, y]$ is not prime, nor radical. The element x is not in I, but x^2 is in I. Note that the set $Z(x^2)$ is an irreducibel hypersurface (see Exercise 7.7.4).

Theorem 7.8.4 (Hilberts Nullstellenzats)**.** *Algebraic sets in $\mathbb{A}^n$ are in one to one correspondance with the set of radical ideal in $\mathbb{C}[x_1, \ldots, x_n]$. Furthermore, we have that an algebraic set is irreducible if and only if its corresponding radical ideal is a prime ideal.*

Corollary 7.8.5. *Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be any proper ideal. Then we have that its radical ideal*

$$\sqrt{I} = \bigcap_{I \subseteq (x_1 - a_1, \ldots, x_n - a_n)} (x_1 - a_1, \ldots, x_n - a_n)$$

where the intersection is taken over the set of maximal ideals containing I.

Proof. The radical of an ideal I is the intersection of all prime ideals containing I, which is a standard fact. The polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ is an example of a Jacobson ring, which means that any intersection of prime ideals is in fact given by the corresponding intersection of maximal ideals. Thus, the radical of I is the intersection of all maximal ideals containing I. By the Nullstellenzats, or the weak version of it, the maximal ideals are all of the form $(x_1 - a_1, \ldots, x_n - a_n)$, with $a_i \in \mathbb{C}$, $i = 1, \ldots, n$. $\square$

Excercise 7.8.6. Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be an ideal. Show that for any integer n we have that

$$Z(I) = Z(I^n).$$

Excercise 7.8.7. Show that for any ideal I we have $Z(I) = Z(\sqrt{I})$.

Excercise 7.8.8. Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be a proper, radical ideal. Assume that I is not prime. Show that there exists elements f and g such that

$$Z(I) = Z(I + f) \cup Z(I + g)$$

is a union of proper subsets. Here $I + f$ means the ideal generated by f and elements of I. Thus if $f_1, \ldots, f_m$ generate I, then $f, f_1, \ldots, f_m$ generate $I + f$.

Excercise 7.8.9. Let $Z \subseteq \mathbb{A}^3$ be the algebraic set defined by the ideal

$$I = (x^2 - yz, xz - x) \subset \mathbb{C}[x, y, z].$$

Show that Z has three components, and describe their corresponding prime ideals.

Excercise 7.8.10. We identify $\mathbb{C}^2$ with $\mathbb{C} \times \mathbb{C}$. Show that the Zariski topology on $\mathbb{C}^2$ is not given by the product topology of $\mathbb{A}^1$ with $\mathbb{A}^1$.

7.9 Polynomial maps

Lemma 7.9.1. *Let $F_i = F_i(x_1, \ldots, x_m)$ be an ordered sequence of n polynomials in m variables ($i = 1, \ldots, n$). The induced map $F \colon \mathbb{A}^m \longrightarrow \mathbb{A}^n$, sending $a = (a_1, \ldots, a_m)$ to $(F_1(a), \ldots, F_n(a))$ is a continuous map.*

Proof. We show that the map is continuous by showing that the inverse image of closed sets are closed. Let $f(y_1, \ldots, y_n)$ be a polynomial in n variables. Let $f \circ F = f(F_1(x), \ldots, F_n(x))$, which is a polynomial in the m variables $x = x_1, \ldots, x_m$. Note that

$$F^{-1}(Z(f)) = \{a \in \mathbb{A}^m \mid (F_1(a), \ldots, F_n(a)) \in Z(f)\},$$

which means that

$$F^{-1}(Z(f)) = \{a \in \mathbb{A}^m \mid f(F_1(a), \ldots, F_n(a)) = 0\} = Z(f \circ F).$$

It follows that if $I \subseteq \mathbb{C}[y_1, \ldots, y_n]$ is an ideal generated by $f_1, \ldots, f_N$, then $F^{-1}(Z(I))$ is the algebraic set given by the ideal $J \subseteq k[x_1, \ldots, x_m]$ generated by $f_1 \circ F, \ldots, f_N \circ F$. $\square$

A map $F\colon \mathbb{A}^m \longrightarrow \mathbb{A}^n$ given by polynomials as in the Lemma, is a polynomial map.

7.10 Maps of affine algebraic sets

If $F\colon \mathbb{A}^m \longrightarrow \mathbb{A}^n$ is a polynomial map that would factorize through an algebraic set $Y \subseteq \mathbb{A}^n$, then we have a map $F\colon \mathbb{A}^n \longrightarrow Y$. Restricting such a map to an algebraic set $X \subseteq \mathbb{A}^m$ gives a map $F\colon X \longrightarrow Y$, which is how we will define a map of algebraic sets.

Remark

Note that our definition of a map between affine algebraic sets, requires an embedding into affine spaces.

Example 7.10.1. The two polynomials $F_1(t) = t^2$ and $F_2(t) = t^3$ determine the polynomial map $F\colon \mathbb{A}^1 \longrightarrow \mathbb{A}^2$. The map F then sends a point $a \mapsto (a^2, a^3)$. Let $Y \subseteq \mathbb{A}^2$ be the curve given by the polynomial $f(x, y) = y^2 - x^3$ in $\mathbb{C}[x, y]$, that is $Y = Z(f)$. For any scalar a we have that the pair (a^2, a^3) is such that $f(a^2, a^3) = 0$. In order words we get an induced map

$$F\colon \mathbb{A}^1 \longrightarrow Y.$$

Definition 7.10.2. *Two affine algebraic sets X and Y are isomorphic, or simply equal, if there exists polynomial maps $F\colon X \longrightarrow Y$ and $G\colon Y \longrightarrow X$ such that $F \circ G = \mathrm{id}$ and $G \circ F = \mathrm{id}$.*

Example 7.10.3. The map $F\colon \mathbb{A}^1 \longrightarrow Y = Z(y^2 - x^3) \subseteq \mathbb{A}^2$ discussed in Example 7.10.1 is a homeomorphism (see Exercise 7.10.5). However, even if the map F gives a homeomorphism between $\mathbb{A}^1$ and the curve $Y \subseteq \mathbb{A}^2$, these two varieties are not considered as equal. The inverse $G\colon Y \longrightarrow \mathbb{A}^1$ to F is not a polynomial map.

Example 7.10.4. Let $p\colon \mathbb{A}^2 \longrightarrow \mathbb{A}^1$ be the projection on the first factor, thus $p(a,b) = a$. Which is a polynomial map. The fiber over a point $a \in \mathbb{A}^1$ is the "vertical" line $Z(x-a)$, where the hypersurface $x - a \in \mathbb{C}[x,y]$. Consider now the polynomial

$$G(x,y) = y^3 + g_1(x)y^2 + g_2(x)y + g_3(x).$$

We get an induced projection map $p_1\colon Z(G) \longrightarrow \mathbb{A}^1$. The fiber over a point $a \in \mathbb{A}^1$ is $Z(x-a, G)$, which is given by

$$g(y) = y^3 + g_1(a)y^2 + g_2(a)y + g_3(a) \in \mathbb{C}[y].$$

The three roots of $g(y)$ are the three points lying above a.

Excercise 7.10.5. Verify that the map $F\colon \mathbb{A}^1 \longrightarrow Y$ in Example 7.10.1 is a homeomorphism, and that the inverse map is not a polynomial map: You first check that the map is bijective, and from Lemma 7.9.1 you have that it is contineous. Then to be able to conclude that the map is a homeomorphism it suffices to verify that the map F gives a bijection between the closed sets of $\mathbb{A}^1$ and Y. The closed sets on Y are the intersection of closed sets on $\mathbb{A}^2$ with Y. The non-trivial closed sets are given by a finite collection of points, and then you have verified that F is a homeomorphism. Then you need to convince yourself that the inverse map G is not a polynomial map.

Excercise 7.10.6. Show that the variety $Z(xy-1) \subseteq \mathbb{A}^2$ is isomorphic to $\mathbb{A}^1 \setminus 0$.

Excercise 7.10.7. Show that a basic open $\mathbb{A}^n \setminus Z(f)$ is an algebraic variety by identifying it with $Z(tf-1) \subseteq \mathbb{A}^{n+1}$.

Chapter 8

Projective varieties

We will define the projective space as a certain quotient space where we identify lines in affine space.

8.1 Projective n-space

Definition 8.1.1. *We let $\mathbb{P}^n$ denote the topological space we obtain by taking the quotient space of $\mathbb{A}^{n+1} \setminus (0, \dots, 0)$ modulo the equivalence relation*

$$(a_0, \dots, a_n) \simeq (\lambda a_0, \dots, \lambda a_n),$$

with non-zero scalars $\lambda \in \mathbb{C}$. The space $\mathbb{P}^n$ is called projective n-space. The equivalence class of a vector $(a_0, \dots, a_n)$ we denote by

$$[a_0 : a_1 : \cdots : a_n].$$

Quotient topology

Recall the notions of quotient topology discussed in Section 1.2.

Excercise 8.1.2. Show that $\mathbb{P}^n$ is a Noetherian space.

8.2 Homogeneous polynomials

Let $S = \mathbb{C}[X_0, \dots, X_n]$ denote the polynomial ring in the variables $X_0, \dots, X_n$. Let $S_d(X_0, \dots, X_n) = S_d$ denote the vector space of degree $d \geq 0$ forms in $X_0, \dots, X_n$. The monomials $\{X_0^{d_0} \cdots X_n^{d_n}\}$ where $d = d_0 + \cdots + d_n$, form a basis for the vector space S_d.

We have the decomposition of vector spaces

$$S = \mathbb{C}[X_0, \dots, X_n] = \bigoplus_{d \geq 0} S_d$$

into homogenous parts. An element $F \in S$ is homogenous, and of degree d, if $F \in S_d$. As the zero polynomial is in S_d for any $d \geq 0$, it has no well-defined degree.

Excercise 8.2.1. Compute the dimension of $S_d(X_0, \dots, X_n)$.

Excercise 8.2.2. The Hilbert series of graded the polynomial ring is the formal expression $H(t) = \sum_{d \geq 0} \dim S_d(X_0, \dots, X_n) t^d$. Show that

$$H(t) = \prod_{i=0}^{n} \frac{1}{1-t}.$$

8.3 Hypersurfaces in projective space

Any homogeneous polynomial $F(X_0, \dots, X_n)$ will satisfy

$$F(\lambda a_0, \dots, \lambda a_n) = \lambda^d F(a_0, \dots, a_n)$$

where d is the degree of $F(X_0, \dots, X_n)$, and $a_0, \dots, a_n$ is any vector in $\mathbb{C}^{n+1}$. Thus, if $a = (a_0, \dots, a_n)$ is an element of $Z(F) \subseteq \mathbb{A}^{n+1}$, then the whole line spanned by a is in $Z(F)$. It follows that the set $Z(F) \setminus (0, \dots, 0)$ in $\mathbb{A}^{n+1} \setminus (0, \dots, 0)$ is invariant with respect to the equivalence relation. In particular we get, by taking the quotient, a closed subset $\overline{Z}(F)$ in $\mathbb{P}^n$. Any homogenous element F of degree $d \geq 1$ will vanish on $(0, \dots, 0)$ in $\mathbb{A}^{n+1}$, that is $(0, \dots, 0) \in Z(F)$. If $Z(F)$ contains other points as well, then we get a non-empty subset $\overline{Z}(F)$ in $\mathbb{P}^n$, and these we refer to as hypersurfaces in projective n-space.

Lectures 4-6

The projective curves discussed in lectures 4-6 are examples of hypersurfaces in the projective plane.

8.4 Homogeneous ideals

An ideal I in the polynomial ring $S = \mathbb{C}[X_0, \dots, X_n]$ is a *homogeneous ideal* if there exist homogenous elements $F_1, \dots, F_m$ that generates the ideal, $I = (F_1, \dots, F_m)$. A homogeneous ideal I can be decomposed as $I = \bigoplus_{d \geq 0} I_d$.

Any homogenous ideal $I \subseteq S$ defines a closed set

$$\overline{Z}(I) = \cap_{i=1}^m \overline{Z}(F_i) \subseteq \mathbb{P}^n,$$

where $F_1, \ldots, F_m$ is a collection of homogenous elements that generate I. Note that any closed subset $Z \subseteq \mathbb{P}^n$ is of the form $Z = \overline{Z}(I)$, for some homogeneous ideal $I \subseteq \mathbb{C}[X_0, \ldots, X_n]$. We refer to the closed sets in $\mathbb{P}^n$ as algebraic sets.

Excercise 8.4.1. Let F be a homogeneous polynomial in $\mathbb{C}[X_0, \ldots, X_n]$, and let I be the homogeneous ideal generated by $X_0F, X_1F, \ldots, X_nF$. Show that

$$\overline{Z}(F) = \overline{Z}(I)$$

as subsets of $\mathbb{P}^n$.

Proposition 8.4.2. *The projective n-space $\mathbb{P}^n$ has a open cover by affine n-spaces. In particular we have identification of varieties*

$$\mathbb{P}^n \setminus \overline{Z}(X_i) = \mathbb{A}^n,$$

for every $i = 0, \ldots, n$, and these open sets cover $\mathbb{P}^n$.

Proof. Let $Z(x_0 - 1)$ be the affine variety in $\mathbb{A}^{n+1}$ where the first coordinate is 1. We will prove that $Z(x_0 - 1)$ can be identified with $U_0 = \mathbb{P}^n \setminus \overline{Z}(X_0)$. The remaining cases are proved similarily. Restricting the projection map $\pi \colon \mathbb{A}^{n+1} \setminus 0 \longrightarrow \mathbb{P}^n$, gives a map

$$\pi_| \colon Z(x_0 - 1) \longrightarrow U_0 = \mathbb{P}^n \setminus \overline{Z}(X_0)$$

that sends $(1, a_1, \ldots, a_n) \mapsto [1 : a_1 : \cdots : a_n]$. We define a map $s \colon U_0 \longrightarrow Z(x_0 - 1)$ by sending

$$s([X_0 : X_1 : \cdots : X_n]) = (1, X_1/X_0, \ldots, X_n/X_0).$$

It is clear that $\pi_|$ is a bijection, with s being its inverse. To conclude that $\pi_|$ is a homeomorphism, we need to show that s is continuous. As closed sets in $\mathbb{A}^{n+1}$ are intersections of hypersurfaces, it suffices to show that the inverse image $s^{-1}(Z(f))$ is closed, for hypersurfaces. We have that $s^{-1}(Z(f)) = \pi_|(Z(f))$. The identification of $\mathbb{A}^n$ with $Z(x_0 - 1) \subseteq \mathbb{A}^{n+1}$, of varieties, is clear. Thus, the polynomial f can be considered as a polynomial in n variables $x_1, \ldots, x_n$. The polynomial

$$F(X_0, \ldots, X_n) = X_0^{\deg f} f(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0})$$

is homogeneous, and by definition $\overline{Z}(F)$ is closed in $\mathbb{P}^n$. We have furthermore that

$$\pi_|(Z(f)) = \overline{Z}(F) \cap U_0,$$

hence closed in U_0. $\square$

Example 8.4.3. Note that setting the first coordinate to 1, and not to say 2, is a choice that corresponds, locally, to a section of the projection map $\mathbb{A}^{n+1} \setminus 0 \longrightarrow \mathbb{P}^n$.

Example 8.4.4. Note that the construction in the proof is a bit ad hoc. We have a homeomorphism identifying $\mathbb{A}^n$ with $\mathbb{P}^n \setminus \overline{Z}(X_i)$, for each i. And we use this identification to give the variety structure on $\mathbb{P}^n$, locally. One could ask why this particular chosen structure was right. It is, and that can be proved using a completely different approach and a different definition of what projective space means.

Glueing

Another way of construction the projective n-space is by glueing, and was discussed in Section 3.2. We recall some of it here. Quite generally, if we have a collection of topological spaces $\{U_i\}_{i \in \mathscr{A}}$ we can glue these together along specified intersections: Assume that we have an inclusion of open subsets $U_{i,j} \subset U_i$, for all indices $i, j \in \mathscr{A}$, and homeomorphisms $\varphi_{i,j} \colon U_{i,j} \longrightarrow U_{j,i}$ that satisfies the co-cycle condition

$$\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$$

when restricted to $U_{i,j} \cap U_{i,k}$, for all $i, j, k \in \mathscr{A}$. Then one can check that the data equals the data of an equivalence relation on the disjoint union $\sqcup U_i$. We can then form the quotient space X by identifying, or glueing, the spaces U_i and U_j together along $U_{i,j} = U_{j,i}$ (identified with $\varphi_{i,j}$).

Excercise 8.4.5. Let $U_i = \mathbb{A}^n$, where $i = 0, \ldots, n$ is fixed. For any $j = 0, \ldots, n$ we let $U_{i,j} = \mathbb{A}^n \setminus Z(x_j)$, and we let

$$\varphi_{i,j} \colon U_{i,j} \longrightarrow U_{j,i}$$

be the map given by the following composition. For the fixed i, we have the map $(a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_{i-1}, 1, a_i, \ldots, a_n)$ putting the 1 on the i'th coordinate, for $i = 0, \ldots, n$. As subsets of $\mathbb{A}^{n+1}$ we have a natural identifications of $U_{i,j}$ with $U_{j,i}$. If $j \leq i$ then the j'th coordinate in $U_{i,j}$ will be invertible, and we consider the map

$$(a_1, \ldots, a_{i-1}, 1, a_i, \ldots, a_n) \mapsto a_j^{-1}(a_1, \ldots, a_{i-1}, 1, a_i, \ldots, a_n)$$

identifying $U_{i,j}$ with $U_{j,i}$. Write up the situation with $j > i$, and show that the identifications satisfy the co-cycle condition. Show furthermore that the glueing of $\sqcup_{i=0}^{n} U_i$ along the identifications $U_{i,j}$ with $U_{j,i}$ gives $\mathbb{P}^n$. In fact the images of U_i in the quotient space are precisely the open sets $\mathbb{A}^n = \mathbb{P}^n \setminus \overline{Z}(X_i)$ given in Proposition 8.4.2

Definition 8.4.6. *An irreducible algebraic set in $\mathbb{P}^n$ is a projective variety.*

Proposition 8.4.7. *Any projective variety $Z \subseteq \mathbb{P}^n$ is given by a homogenous ideal I that is a prime ideal in $\mathbb{C}[X_0, \ldots, X_n]$.*

Proof. Let $\pi \colon \mathbb{A}^{n+1} \setminus 0 \longrightarrow \mathbb{P}^n$ be the quotient map, where $0 = (0, \ldots, 0)$. A set $X \subseteq \mathbb{A}^{n+1}$ is irreducible if and only if $X_0 = X \setminus 0$ is irreducible or empty in $\mathbb{A}^{n+1} \setminus 0$. If X_0 is irreducible, then $\pi(X_0)$ is also irreducible since the quotient map π is contineous. Thus we see that for any prime ideal $I \subseteq \mathbb{C}[X_0, \ldots, X_n]$ which is also homogeneous, we have that $\overline{Z}(I)$ is irreducible (or empty). Conversely, let $Z \subseteq \mathbb{P}^n$ be an irreducible set. Then $Z = \overline{Z}(I)$ for some homogeneous radical ideal $I \subseteq \mathbb{C}[X_0, \ldots, X_n]$. If I is not prime, then there exists elements F and G not in I but where $FG \in I$. One checks that we can assume the elements F and G to be homogeneous elements. But, then we have, as in Exercise 7.8.8,

$$Z(I + F) \cup Z(I + G) = Z(I),$$

with proper subsets $Z(I + F) \subset Z(I)$, and $Z(I + G) \subset Z(I)$. It follows that $\overline{Z}(I) = \overline{Z}(I + F) \cup \overline{Z}(I + G)$ is the union of two proper closed subsets. Hence $\overline{Z}(I)$ was not irreducible after all. We can therefore conclude that the ideal I was prime. □

Excercise 8.4.8. The projective set C in $\mathbb{P}^3$ given by the homogeneous ideal generated by

$$XW - YZ, \quad XZ - Y^2 \quad \text{and} \quad YW - Z^2$$

in $\mathbb{C}[X, Y, Z, W]$ is called the twisted cubic. If we let $[t : u]$ be projecive coordinates for the projective line, then the twisted cubic is the set of points in $\mathbb{P}^3$ of the form $[t^3 : t^2u : tu^2 : u^3]$.

Excercise 8.4.9. The intersection of varieties is not always a variety. Consider the two surfaces Q_1 and Q_2 in $\mathbb{P}^3$, given by the ideals generated by the quadratic polynomials

$$F = Z^2 - YW \quad \text{and} \quad G = XY - ZW$$

in $\mathbb{C}[X, Y, Z, W]$. Show that both $Q_1 = Z(F)$ and $Q_2 = Z(G)$ are varieties. Identify furthermore their intersection $Q_1 \cap Q_2$ as the union of a twisted cubic, and a line. In fact, we have that the intersection is given as the union of the line $Z(X, W)$ and the twisted cubic $C = Z(F, G, Y^2 - XZ)$ (see Exercise 8.4.8).

Excercise 8.4.10. The ideal generated by the union does not always describe the intersection. Let C be the curve given by the ideal $I(C) = (X^2 - YZ)$, and let L be the line $I(L) = (Y)$. Show that the intersection $C \cap L$ is a point $P \in \mathbb{P}^2$. Compute the homogeneous prime ideal $I(P)$ corresponding to the point P, and deduce that $I(C) + I(L) \neq I(P)$. Draw a picture to explain the situation.

Excercise 8.4.11. Let $P = [1 : 0 : 0 : 0]$ and $Q = [0 : 1 : 0 : 0]$ be points in $\mathbb{P}^3$. There is a unique line L passing through P and Q. A parametric description of the line is the set $tP + uQ$, with projective parameters $[t : u] \in \mathbb{P}^1$. Describe the homogenous prime ideal $I \subseteq \mathbb{C}[x, y, z, w]$ defining L.

Excercise 8.4.12. Consider the quadratic surface $Q \subset \mathbb{P}^3$ given by the polynomial $XY - ZW$ in $\mathbb{C}[X, Y, Z, W]$. Show that Q contains two families of lines L_P and N_P parametrized by points $P = [t : u] \in \mathbb{P}^1$, with the following property.

$$L_P \cap L_{P'} = N_P \cap N_{P'} = \emptyset \quad \text{if} \quad P \neq P',$$

and

$$L_P \cap N_{P'} = \text{a point} \quad \text{for all } P, P'.$$

Hint, use that $XY - ZW$ is the determinant of the matrix

$$\begin{bmatrix} X & Z \\ W & Y \end{bmatrix}.$$

Chapter 9

Maps of projective varieties

9.1 Quasi-projective varieties

Before we continue with what should be a map between projective varieties, it turns out that it is convenient to define the notion of quasi-projective sets.

Definition 9.1.1. *A set $X \subseteq \mathbb{P}^n$ is quasi-projective if it is locally closed; meaning that there exists a (algebraic) closed set $Z \subseteq \mathbb{P}^n$ containing $X \subseteq Z$ as an open subset.*

Example 9.1.2. Any algebraic set $X \subseteq \mathbb{P}^n$ is quasi-projective since X is open in itself, and an algebraic set is by definition closed in $\mathbb{P}^n$. Any open subset $U \subseteq X$ of an algebraic set $X \subseteq \mathbb{P}^n$ is by defintion quasi-projective. In particular the affine n-space $\mathbb{A}^n$ is quasi-projective since $\mathbb{A}^n = \mathbb{P}^n \setminus \overline{Z}(X_0)$ is open in $\mathbb{P}^n$. Any open subset $U \subseteq \mathbb{A}^n$ is quasi-projective since it will also be open in $\mathbb{P}^n$.

Excercise 9.1.3. Show that any affine algebraic set $X = Z(I) \subseteq \mathbb{A}^n$ is quasi-projective by showing that there is a homogenous ideal $J \subseteq \mathbb{C}[X_0, \ldots, X_n]$ such that

$$X = \overline{Z}(J) \cap \mathbb{P}^n \setminus \overline{Z}(X_0).$$

Then X will be open in $\overline{Z}(J)$. I think you will prove something as: For any $f \in \mathbb{C}[t_1, \ldots, t_n]$, let $h(f) = x_0^{\deg f} f(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0})$. Then $h(f)$ is a homogeneous element of $\mathbb{C}[X_0, \ldots, X_n]$. For any homogeneous polynomial $F \in \mathbb{C}[X_0, \ldots, X_n]$ we let $d(F) = F(1, t_1, \ldots, t_n)$. We have that

$$d(h(f)) = f \quad \text{and} \quad h(d(F)) = x_0^N F \quad \text{some} \quad N \geq 0.$$

Excercise 9.1.4. Let $Y \subseteq \mathbb{A}^n$ be an algebraic set, and let $I(Y)$ be an ideal defining Y. We view Y as a subset of $\mathbb{P}^n$, by identifying $\mathbb{A}^n = \mathbb{P}^n \setminus Z(X_0)$. The closure $\overline{Y}$ is the smallest closed subset containing Y, and is then by definition an algebraic set. Show that the $h(I(Y))$, obtained by applying the h function in Exercise 9.1.3 to all elements of $I(Y)$, generates the homogeneous ideal $I(\overline{Y})$ describing $\overline{Y}$.

Excercise 9.1.5. Let $Y \subseteq \mathbb{A}^3$ be the twisted cubic, that is the image of $\mathbb{A}^1 \longrightarrow \mathbb{A}^3$ by the map $t \mapsto (t, t^2, t^3)$. Use this example to show that if $f_1, \ldots, f_r$ generates an ideal $I(Y)$, then $h(f_1), \ldots, h(f_r)$ does not generate the ideal $I(\overline{Y})$ of its closure in projective space (see Exercise 9.1.4).

The affine varities are the building blocks

The notion of quasi-projective sets includes both projective and affine algebraic sets. Note that sets as $\mathbb{A}^2 \setminus (0,0)$ are also quasi-projective, these are however not affine algebraic sets. At first glance it now would appear as the notion of quasi-projective varieties is to genereous, forcing us to also deal with sets which are not build up via affine charts. However, as we saw in Exercise 7.10.7 the basic open sets are affine algebraic sets. So indeed $\mathbb{A}^2 \setminus = \mathbb{A}^2 \setminus Z(x) \cup \mathbb{A}^2 \setminus Z(y)$ is the union of varities.

9.2 Regular maps

Let $F_0, \ldots, F_n$ be $n+1$ homogenous polynomials each of the same degree, in $m+1$ variables. Having these polynomials we get a polynomial map

$$F = (F_0, \ldots, F_n) \colon \mathbb{A}^{m+1} \longrightarrow \mathbb{A}^{n+1}. \tag{9.1}$$

The fact that the polynomials all have the same degree guarantees that the map of affine spaces respects the equivalence classes defining the projective space. In order to get an induced map of projective spaces we need furthermore that the common zero of the polynomials is the origo only.

Definition 9.2.1. *Let $X \subseteq \mathbb{P}^m$ be a quasi-projective set. A regular map $F \colon X \longrightarrow \mathbb{P}^n$ is a sequence $F = (F_0, \ldots, F_n)$ of homogeneous polynomials in $m+1$-variables, all of same degree, and where the polynomials $F_0, \ldots, F_n$ do not simultaneously vanish on X. The last condition is that*

$$X \cap \overline{Z}(F_0) \cap \cdots \cap \overline{Z}(F_n) = \emptyset.$$

Example 9.2.2. Consider the polynomial map $\mathbb{A}^3 \longrightarrow \mathbb{A}^2$ given by the two linear polynomials $F_0 = X$ and $F_1 = Y - Z$. So the polynomial map sends

$$(a, b, c) \mapsto (a, b - c).$$

This will respect the equivalence classes defining the projective space, but will not induce a map from the projective plane to the projective line. This is because the line $(0, t, t)$ is sent to $(0, 0)$. In order words we get an induced map

$$\mathbb{P}^2 \setminus [0 : 1 : 1] \longrightarrow \mathbb{P}^1$$

defined on the complement of a point in the projective plane. The regular map defined above can not be extended to a regular map defined on the whole of projective plane.

Example 9.2.3. Let $Q \subset \mathbb{P}^2$ be the quadratic curve given by the equation $X^2 + Y^2 + Z^2$. As the point $[0 : 1 : 1]$ is not on Q, the map described in the previous example gives a map $F \colon Q \longrightarrow \mathbb{P}^1$ that takes

$$[X : Y : Z] \mapsto [X : Y - Z].$$

So the map $F \colon Q \longrightarrow \mathbb{P}^1$ is given by the homogenous polynomials $F_0 = X$ and $F_1 = Y - Z$ in the variables X, Y, Z.

9.3 Maps of projective varieties

We are now ready to define what we mean with a map of projective varieties.

Definition 9.3.1. *Let $X \subseteq \mathbb{P}^m$ be a quasi-projective algebraic set. A map $F \colon X \longrightarrow \mathbb{P}^n$ is a finite collection of regular maps $F_i \colon X_i \longrightarrow \mathbb{P}^n$ ($i = 1, \ldots, N$), where*

(1) we have that $X_i \subseteq X$ is an open subset, for each $i = 1, \ldots, N$,

(2) the opens cover $X = \cup_{i=1}^N X_i$,

(3) restricted to intersections $X_i \cap X_j$ the regular maps F_i and F_j agree, for all $i, j \in 1, \ldots, N$.

Furthermore, if $X \subseteq \mathbb{P}^m$ and $Y \subseteq \mathbb{P}^n$ are two projective sets, then a map $F \colon X \longrightarrow Y$ is a map $F \colon X \longrightarrow \mathbb{P}^n$ that factors via the inclusion $Y \subseteq \mathbb{P}^n$. Note that a regular map $F \colon X \longrightarrow \mathbb{P}^n$ is in particular a map of projective varieties.

Example 9.3.2. Let $C \subset \mathbb{P}^2$ be the quadratic curve given by the equation $X^2 - Y^2 + Z^2$. The map in Example 9.2.2 induces, by restriction, a map

$$F_1 \colon C \setminus [0 : 1 : 1] \longrightarrow \mathbb{P}^1$$

that sends $[X : Y : Z]$ to $[X : Y - Z]$. In a similar way we have a map $F_2 \colon C \setminus [0 : -1 : 1] \longrightarrow \mathbb{P}^1$ that sends

$$[X : Y : Z] \mapsto [Y + Z : X].$$

There is nothing wrong with these two maps F_1 and F_2, both being given by polynomials, but none of these two can be extended to a polynomial map from the whole curve. However, together they describe a map. Note that when $X \neq 0$ we have

$$[X : Y - Z] = [1 : \frac{Y - Z}{X}] = [1 : \frac{Y^2 - Z^2}{X(Y + Z)}] = [Y + Z : \frac{X^2}{X}].$$

In other words, the two regular maps F_1 and F_2 are equal when restricted to

$$C \setminus [0 : 1 : 1] \cup [0 : -1 : 1].$$

Therefore, together the two maps F_1 and F_2 describe a map from the union

$$F \colon C \longrightarrow \mathbb{P}^1.$$

It should be clear from the example above that we need to accept that the maps of projective varieties can only be locally defined by polynomials. However, when one have defined the maps locally then one can wonder how small or local these defining charts need to be. The situation is not that bad, as the following statement shows.

Proposition 9.3.3. *A map $F \colon X \longrightarrow Y$ of affine algebraic sets is given by a polynomial map.*

Proof. We do not prove it here, but we want to point out the following. Any affine algebraic set X is a quasi-projective set (Exercise 9.1.3). Hence it makes sense to talk about maps of affine algebraic sets, considered as quasi-projective sets. If $X \subseteq \mathbb{A}^n$, then $X = Z(I)$ for some ideal $I \subseteq \mathbb{C}[x_0, \dots, x_n]$. One shows that the maps $X \longrightarrow \mathbb{A}^m$ are simply m elements in the quotient ring $\mathbb{C}[x_0, \dots, x_n]/I$. And it follows that maps of affine algebraic sets are the same as polynomial maps. $\square$

Excercise 9.3.4. What should two isomorphic projective varieties mean? You probably can imagine several reasonable definitions, and these are probably all correct.

Excercise 9.3.5. The degree of a map of projective curves is the number of points in a generic fiber. What is the degree of the map $F \colon Q \longrightarrow \mathbb{P}^1$ given in Example 9.3.2

9.4 The Veronese map

Let $d \geq 1$ be a given integer. For each integer $n \geq 0$ we will describe the Veronese map

$$v_d \colon \mathbb{P}^n \longrightarrow \mathbb{P}^N$$

where $N+1$ is the number of monomials in $n+1$ variables, in degree d, i.e. $N+1$ is the dimension of $S_d(X_0, \ldots, X_n)$. Define the set

$$\mathscr{D}_d^n = \mathscr{D} = \{\underline{d} = (d_0, \ldots, d_n) \in \mathbb{N}^{n+1} \mid d_0 + \cdots + d_n = d\}. \tag{9.2}$$

Then clearly the monomials in $X_0, \ldots, X_n$ of degree d correspond, naturally, to the elements of $\mathscr{D}$. If $\underline{d} \in \mathscr{D}$ then the corresponding monomial $X^{\underline{d}} = X_0^{d_0} \cdots X_n^{d_n}$.
We have the polynomial map $\mathbb{A}^{n+1} \longrightarrow \mathbb{A}^{N+1}$, that sends a vector $(a_0, \ldots, a_n)$ to $(a^{\underline{d}})_{\underline{d} \in \mathscr{D}}$. The map respects the equivalence relation defining the projective spaces, since the monomials are all homogeneous of the same degree, and hence gives an induced map $v_d \colon \mathbb{P}^n \longrightarrow \mathbb{P}^N$ taking

$$[X_0 : \cdots : X_n] \mapsto [X^{\underline{d}}]_{\underline{d} \in \mathscr{D}}.$$

Proposition 9.4.1. *Let $d \geq 1$ and $n \geq 0$ be two given integers, and let $\mathbb{C}[X_{\mathscr{D}}]$ be the graded polynomial ring with variables indexed by the set $\mathscr{D}$ given in 9.2. The Veronese map $v_d \colon \mathbb{P}^n \longrightarrow \mathbb{P}^N$ identifies $\mathbb{P}^n$ with the Veronese variety $V_{n,d}$ given by the homogeneous ideal*

$$I = (X_{\underline{d}} X_{\underline{d}'} - X_{\underline{e}} X_{\underline{e}'}) \subset \mathbb{C}[X_{\mathscr{D}}],$$

where the quadratic equations $X_{\underline{d}} X_{\underline{d}'} - X_{\underline{e}} X_{\underline{e}'}$ is formed for every quadruple $\underline{d}, \underline{d}', \underline{e}, \underline{e}'$ in $\mathscr{D}$ such that $\underline{d} + \underline{d}' = \underline{e} + \underline{e}'$.

Proof. Let $\underline{i}$ denote the element in $\mathscr{D}$ that has the number d in coordinate i, and zero elsewhere (with $i = 0, \ldots, n$). Thus $\underline{i}$ corresponds to the monomial X_i^d. For each $i = 1, 0, \ldots, n$ we let

$$U_i = \overline{Z}(I) \cap \mathbb{P}^N \setminus \overline{Z}(X_{\underline{i}}).$$

Then one checks that the open sets $U_0, \ldots, U_n$ cover the algebraic set $\overline{Z}(I)$. We will next indicate how maps $G_i \colon U_i \longrightarrow \mathbb{P}^n$ are defined by simply giving the definition of G_0. Let $d(i) \in \mathscr{D}$ be the element $d(i) = (d-1, 0, \ldots, 1, \ldots, 0)$, where the 1 appears in coordinate i (and $i = 1, \ldots, n$). We define the map $G_0 \colon U_0 \longrightarrow \mathbb{P}^n$ by

$$G_0([X_{\underline{d}}]) = [X_{\underline{0}} : X_{d(1)} : \cdots : X_{d(n)}].$$

One checks that the similarily defined maps G_i and G_j coincide on $U_i \cap U_j$ (for all pairs i, j), and hence we have a map

$$G\colon \overline{Z}(I) \longrightarrow \mathbb{P}^n.$$

Clearly the maps G_i will factor via the open inclusion $\mathbb{P}^n \setminus \overline{Z}(X_i)$. Let F_i be the restriction of the Veronese map v_d to the open subset $\mathbb{P}^n \setminus \overline{Z}(X_i)$. Note that since $X_0 \neq 0$ on U_0, we have that

$$G_0([X_{\underline{d}}]) = X_0^d[1\colon \frac{X_1}{X_0}\colon \cdots \colon \frac{X_n}{X_0}].$$

One verifies that G_i is the inverse of F_i, so G is the inverse of F, and you have proven that the Veronese map is an isomorphism of projective varieties. $\square$

Example 9.4.2. The Veronese surface is given by the map $v_2\colon \mathbb{P}^2 \longrightarrow \mathbb{P}^5$ that sends

$$[X : Y : Z] \mapsto [X^2 : XY : XZ : Y^2 : YZ : Z^2].$$

The defining equations for the Veronese surface is given by the quadratic polynomials in $\mathbb{C}[Z_0, \dots, Z_5]$, given as the (2×2)-minors of

$$\begin{bmatrix} Z_0 & Z_3 & Z_4 \\ Z_3 & Z_1 & Z_5 \\ Z_4 & Z_5 & Z_2 \end{bmatrix}.$$

This is a neat way of describing all two pairs (d, d') and (e, e') of vectors, among $(2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)$ that have the same sum $d + d' = e + e'$. There are nine (2×2)-minors, but since the matrix is symmetric, only six of them are relevant.

Excercise 9.4.3. Let $V \subseteq \mathbb{P}^5$ denote the Veronese surface, given in Example 9.4.2. Show that for any two points P and Q on V there exists a conic curve C on V passing through the points P and Q. The curve C will be an embedding of the projective line $\mathbb{P}^1$.

9.5 Veronese subvarieties

As the projective n-space $\mathbb{P}^n$ is identified with the Veronese variety $V_{d,n} \subseteq \mathbb{P}^N$, it means in particular that subvarities $\overline{Z}(I)$ in $\mathbb{P}^n$ are identified with subvarieties in $V_{d,n}$. We will look a bit closer att that correspondance. Thus let $d \geq 1$ and $n \geq 0$ be fixed integers. Note that a homogenous element $F \in \mathbb{C}[X_0, \dots, X_n]$ of degree $d \cdot e$, naturally can be viewed as a polynomial in the monomials of degree d.

Furthermore, if $G \in \mathbb{C}[X_0, \dots, X_n]$ is homogenous of degree e, then $I = (X_0G, \dots, X_nG)$ is homogeneous of degree one more, $e+1$, and we have that $\overline{Z}(G) = \overline{Z}(I)$ (see Exercise 8.4.1). Hence, if $I \subseteq \mathbb{C}[X_0, \dots, X_n]$ is a homogeneous ideal generated by elements of degree $\leq e$, then we can form the ideal I' that is generated in degree equal to $d \cdot e$, and where $\overline{Z}(I) = \overline{Z}(I')$. However, the generators of I' correspond to degree e polynomials in the degree d monomials in $\mathbb{P}^N$, wherein the Veronese variety $V_{d,n}$ is.

Example 9.5.1. We have the Veronese map $v_2 \colon \mathbb{P}^2 \longrightarrow \mathbb{P}^5$, identifying the projective plane with the Veronese surfaces $V_{2,2}$. In the plane we have the cubic curve

$$C = \overline{Z}(X^3 + Y^3 + Z^3) \subset \mathbb{P}^2.$$

We write C as the intersection of the three quartics

$$X^4 + XY^3 + XZ^3, \quad X^3Y + Y^4 + YZ^3 \quad \text{and} \quad X^3Z + Y^3Z + Z^4.$$

These three polynomials of degree $2 \cdot 2$, correspond to the following degree 2 polynomials in $\mathbb{C}[Z_0, \dots, Z_n]$,

$$Z_0^2 + Z_1Z_3 + Z_2Z_5, \quad Z_0Z_1 + Z_3^2 + Z_4Z_5 \quad \text{and} \quad Z_0Z_2 + Z_3Z_4 + Z_5^2.$$

Intersecting the zero set of these three polynomials with the Veronese surface $V_{2,2}$ gives $v_2(C) \subseteq \mathbb{P}^5$.

Excercise 9.5.2. Show that any projective variety is isomorphic to an intersection of a Veronese variety ($v_d(\mathbb{P}^n)$ for some n, d), with a linear space.

9.6 The Segre maps

We let $\mathbb{P}^m \times \mathbb{P}^n$ denote the set of ordered pairs of points $([X],[Y])$, with $[X]$ in $\mathbb{P}^m$ and $[Y]$ in $\mathbb{P}^n$. The Segre map

$$\sigma \colon \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}$$

is defined by sending

$$([X_0 : \cdots : X_m], [Y_0 : \cdots : Y_n]) \mapsto [\cdots : X_iY_j : \cdots].$$

Proposition 9.6.1. *The image of the Segre map $\sigma \colon \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}$ is the Segre varietey $\Sigma_{m,n}$ defined by the homogeneous ideal*

$$I = (Z_{i,j}Z_{k,l} - Z_{i,l}Z_{k,j}) \subset \mathbb{C}[Z_{i,j}]_{\substack{0 \leq i \leq m \\ 0 \leq j \leq n}}.$$

Proof. □

Example 9.6.2. The image of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ by the Segre map is the quadratic surface given by

$$XW - YZ \in \mathbb{C}[X, Y, Z, W].$$

Product varieties

Via the Segre map we identify $\Sigma_{m,n}$ with the product $\mathbb{P}^m \times \mathbb{P}^n$, and in particular we can give the product the structure of a variety. This is of course an ad hoc construction. We can define and talk about products of varieties, even if we do not define that concept here.

9.7 Bi-homogenous forms

A polynomial $F(X,Y) \in \mathbb{C}[X_0,\dots,X_n,Y_0,\dots,Y_m]$ in two set of variables X and Y, is bihomogenous of degree (d,e), if of the form

$$F(X,Y) = \sum_{|\alpha|=d,\, |\beta|=e} c_{\alpha,\beta} X^\alpha Y^\beta,$$

where $c_{\alpha,\beta}$ are complex numbers, and $\alpha = (\alpha_0,\dots,\alpha_n)$ is multi-index notation with $\mid \alpha \mid = \sum_{i=0}^n \alpha_i$. And similarly with $\beta = (\beta_0,\dots,\beta_m)$. Note that any bigraded polynomial $F(X,Y)$ gives a well-defined pair of closed subsets $Z(F) \subseteq \mathbb{P}^n \times \mathbb{P}^m$.

Example 9.7.1. Let $C \subseteq \mathbb{P}^3$ be the twisted cubic, defined by the quadratic polynomials

$$F = XW - YZ, \quad G = XZ - Y^2 \quad \text{and} \quad H = YW - Z^2$$

in $\mathbb{C}[X,Y,Z,W]$. In $\mathbb{P}^3$ we have the Segree surface Σ being the image of $\mathbb{P}^1 \times \mathbb{P}^1$ (see Example 9.6.2). The surface Σ is cut out by $F = XW - YZ$, and in particular we have that our curve $C \subset \Sigma$. As Σ is $\mathbb{P}^1 \times \mathbb{P}^1$ we can relate algebraic subvarieties of Σ with bihomogenous ideals in $\mathbb{C}[X_0,X_1,Y_0,Y_1]$. The polynomial G restricted to $\mathbb{P}^1 \times \mathbb{P}^1$ is then the polynomial

$$X_0X_1Y^2 - X_0Y_1^2 = X_0 \cdot g \quad \text{where} \quad g = X_1Y_0^2 - X_0Y_1^2.$$

The polynomial g is bigraded of degree $(1,2)$. The restriction of $Z(G)$ to Σ is then the union of a line $Z(X_0)$ and the curve $Z(g)$. Similarily, the polynomial H becomes

$$X_0X_1Y_1^2 - X_1^2Y_0^2 = -X_1 g,$$

so $Z(H) \cap \Sigma$ is a line union the curve $Z(g)$. In other words the twisted curve C is given by a single polynomial g, of bidegree $(1,2)$ over the Veronese surface Σ.

Excercise 9.7.2. Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be projective varieties, given by homogeneous ideals

$$I_X \subseteq \mathbb{C}[X_0, \ldots, X_n] \quad \text{and} \quad I_Y \subseteq \mathbb{C}[Y_0, \ldots, Y_m].$$

We define the maps

$$i \colon X \longrightarrow \mathbb{P}^{m+n+1} \quad \text{and} \quad j \colon Y \longrightarrow \mathbb{P}^{m+n+1}$$

by $i(x) = [x : 0]$, and $j(y) = [0 : y]$, for any point $x \in X$ and $y \in Y$, and where 0 denotes the sequence of $m + 1$, respectively $n + 1$, zeros. Show that i identifies X with $i(X)$, and describe the ideal $I'_X \subseteq \mathbb{C}[X, Y]$ describing $i(X) \subseteq \mathbb{P}^{m+n+1}$. Show that

$$i(X) \cap j(Y) = \emptyset.$$

Let finally $I^e_X \subseteq \mathbb{C}[X, Y]$ denote the ideal generated by I_X, and similarily with I^e_Y. Show that the ideal $I^e_X + I^e_Y$ describes the join $J(X, Y) \subseteq \mathbb{P}^{m+n+1}$ consisting of all points on lines $L(x, y)$ between a point $x \in i(X)$ and a point $y \in j(Y)$.

Bibliography

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