Algebra and geometry through projective spaces
Problems on toric geometry

due: Monday, May 15. Please send a file.pdf to dirocco@kth.se.

Problem 1. Consider a minimal hyperplane description of a lattice polytopes P. In other words let $P = \bigcap_{i=1}^{s} H^{+}_{\xi_i,b_i}$ where s is the the minimum number of half-spaces necessary to cut out P. Show that P has s facets and that the vectors ξ_i are normal vectors to the associated facet. Moreover show that the pairs (ξ_i, b_i) are uniquely determined up to enumeration (the vectors ξ_i are unique up to positive scalar factors).

Problem 2. Classify, up to affine equivalence, all the smooth polygons containing at most 8 lattice points.

Problem 3. Let P be a smooth polytope. Prove that $X_v \cong \mathbb{C}^n$ for every vertex v.

Problem 4. Recall that $kP = \{m_1 + \ldots + m_k$ s.t. $m_i \in P\}$ and that if $P_1 \subset \mathbb{R}^n, P_2 \subset \mathbb{R}^t$ then $P_1 \times P_2 = \{(m,n)$ s.t. $m \in P_1, m \in P_2\} \subset \mathbb{R}^n \times \mathbb{R}^t$ is a polytope of dimension $\dim(P_1) + \dim(P_2)$ and whose faces are products of faces of resp. polytopes.

(1) Describe the faces of the polytope $P = \Delta_1 \times 2\Delta_2$.
(2) Is P smooth?
(3) Describe the toric variety X_P as union of affine patches.
(4) Describe the induced map $X_P \to \mathbb{P}^{11}$.

Problem 5. A rational normal curve of degree d is defined as the image of the degree d Segre embedding of $\mathbb{P}^1$:

$$\mathbb{P}^1 \to \mathbb{P}^d \ \ (x_0 : x_1) \mapsto (x_0^d : x_0^{d-1}x_1 : x_0^{d-2}x_1^2 : \ldots : x_0 x_1^{d-1} : x_1^d)$$

Let P be a lattice polytope. Show that for every edge $L \subset P$, the toric variety $V(L)$ is smooth and isomorphic to a rational normal curve. What is the degree of such rational curve?

Problem 6. Let $a_0, \ldots, a_n$ be coprime positive integers. Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ given by:

$$t \cdot (x_1, \ldots, x_n) = (t^{a_0}x_0, \ldots, t^{a_n}x_n) = \mathbb{P}(a_0, \ldots, a_n).$$

The quotient $(\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$ exists and it is called the **weighted projective space with weights** $a_0, \ldots, a_n$.

(1) In which sense is this a generalisation of $\mathbb{P}^n$?
(2) We say that a polynomial $p(x) = \sum_\alpha c_\alpha x^\alpha \in \mathbb{C}[x_0, x_1, x_2, \ldots, x_n]$ is $(a_0, a_1, \ldots, a_n)-$homogeneous of weighted degree s if every monomial x^α satisfies $\alpha \cdot (a_0, \ldots, a_n) = s$. Show that $f = 0$ is a well defined equation on $\mathbb{P}(a_0, \ldots, a_n)$ if and only if f is $(a_0, a_1, \ldots, a_n)-$homogeneous.
(3) Consider $\mathbb{P}(1,1,d)$. Show that the map $\mathbb{P}(1,1,d) \to \mathbb{P}^d + 1$ defined by $(x_0, x_1, x_2) \to (x_0^d, x_x^{d-1}x_1, \ldots, x_o x_1^{d-1}, x_1^d, x_2)$ is well defined.

(4) Show that $\mathbb{P}(1, 1, d)$ is a projective toric variety.
(5) Construct the polytope associated to $\mathbb{P}(1, 1, d)$.
(6) (*)[bonus point] Can you show (4) and (5) for any $\mathbb{P}(a_0, \ldots, a_n)$?