KTH Teknikvetenskap

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# Chapter 10

# projective toric varieties and polytopes: definitions

## 10.1 Introduction

Tori varieties are algebraic varieties related to the study of **sparse polynomials.** A polynomial is said to be sparse if it only contains prescribed monomials.

Let $A = \{m_0, \dots, m_d\} \subset \mathbb{Z}^n$ be a finite subset of integer points. We will use the multi-exponential notation:

$$x^a = x_1^{a_1} \cdot x_2^{a_2} \cdots x_n^{a_n} \text{ where } x = (x_1, \dots, x_n) \text{ and } a = (a_1, \dots, a_n) \in \mathbb{Z}^n$$

Sparse polynomials of type $A$ are polynomials in $n$ variables of type:

$$p(x) = \sum_{a \in A} c_a x^a$$

For example if $A = \{(i,j) \in \mathbb{Z}_+^2 \text{ such that } i + j \leq k\}$ then the polynomials of type $A$ are all possible polynomials of degree up to $k$.

Toric varieties admit equivalent definitions arising naturally in many mathematical areas such as: Algebraic Geometry, Symplectic Geometry, Combinatorics, Statistics, Theoretical Physics etc.

We will present here an approach coming from Convex geometry and will see that toric varieties represent a natural generalization of projective spaces.

There are two main features we will try to emphasise:

(1) toric varieties, $X$, are prescribed by **sparse polynomials**, in the sense that they are mapped in projective space via these pre-assigned monomials, whose exponents span an integral polytope **polytope** $P_X$. You

can think at a parabola parametrized locally by $t \mapsto (t, t^2)$. The monomials are prescribed by the points $1, 2 \in \mathbb{Z}$. The polytope spanned by these points is a segment of length 1, $[1, 2]$. Discrete data $A$ (i.e. points in $\mathbb{Z}^n$ ) gives rise to a polytope $P_A$ and in turn to a torc variety $X_A$ allowing a geometric analysis of the original data. This turns out to be very useful in Statistics or Bio-analysis for example.

(2) Toric varieties are defined by **binomial ideals**, i.e. ideals generated by polynomials consisting of two monomials: $x^u - x^v$. In the example of the parabola all the points in the image are zeroes of the binomial: $y - x^2$. This feature is particularly useful in integer programming when one wants to find a vertex (of the associated polytope) that minimises a certain (cost) function.

## 10.2 Recap example

Consider the ideal $(x^3 - y^2) \in \mathbb{C}[x, y]$.

(a) The generating polynomial is irreducible and thus the corresponding affine variety $X = Z(x^3 - y^2) \subset \mathbb{C}^2$ is an **irreducible affine variety**.

(b) Consider now the **algebraic torus** $\mathbb{C}^* = \mathbb{C} \setminus \{0\} \subset \mathbb{C}$. Notice that $\mathbb{C}^* = \mathbb{C} \setminus Z(x)$, a Zariski-open subset of $\mathbb{C}$. Consider now the map $\phi : \mathbb{C}^* \to \mathbb{C}^2$ defined as $\phi(t) = (t^2, t^3)$. Observe that $Im(\phi) \subseteq X$ and that $\psi : Im(\phi) \to \mathbb{C}^*$ defined as $\phi(x, y) = (y/x)$ is an inverse. It follows that $\mathbb{C}^* \cong Im(\phi)$, i.e. $\mathbb{C}^* \subset X$.

(c) The open set $\mathbb{C}^*$ is also a multiplicative group. We can define a **group action** on $X$ as follows:

$$\mathbb{C}^* \times X \to X, (t, (x, y)) \mapsto (t^2 x, t^3 y).$$

Notice that, by definition, the action restricted to $\mathbb{C}^* \subset X$ is the multiplication in the group.

We will call such a variety, i.e. a variety satisfying (a), (b) and (c), an **affine toric variety**

## 10.3 Algebraic tori

**Definition 10.3.1.** *A* **linear algebraic group** *is a Zariski-open set $G$ having the structure of a group and such that the multiplication map and the* *inverse map:*

$$m : G \times G \to G, i : G \to G$$

*are morphisms of affine varieties.*

*Let $G, G'$ be two linear algebraic groups, a* **morphism** *$G \to G'$ of linear algebraic groups is a map which is a morphism of affine varieties and a homomorphism of groups.*

We will indicate the SET of such morphisms with $Hom_{AG}(G, G')$.

*Excercise* 10.3.2. Show that when $G, G'$ are abelian $Hom_{AG}(G, G')$ is an abelian group.

*Example* 10.3.3. The classical examples of algebraic groups are:$(\mathbb{C}^*)^n, GL_n, SL_n$.

**Definition 10.3.4.** *An $n$-dimensional algebraic torus is a Zariski-open set $T$, isomorphic to $(\mathbb{C}^*)^n$.*

An algebraic torus is a group, with the group operation that makes the isomorphism (of affine varieties) a group-homomorphism. Hence an algebraic torus is a linear algebraic group.

From now on we will drop the adjective algebraic in algebraic torus.

**Definition 10.3.5.** *Let $T$ be a torus.*

- *An element of the abelian group $Hom_{AG}(\mathbb{C}^*, T)$ is called a one parameter subgroup of $T$.*
- *An element of the abelian group $Hom_{AG}(T, \mathbb{C}^*)$ is called a character of $T$.*

**Lemma 10.3.6.** *Let $T \cong (\mathbb{C}^*)^n$ be a torus.*

$$Hom_{AG}(T, \mathbb{C}^*) \cong \mathbb{Z}^n.$$

*Proof.* Because $Hom_{AG}(T, \mathbb{C}^*) \cong (Hom_{AG}(\mathbb{C}^*, \mathbb{C}^*))^n$ it suffices to prove that $Hom_{AG}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{Z}$. Let $F : \mathbb{C}^* \to \mathbb{C}^*$ be an element of $Hom_{AG}(\mathbb{C}^*, \mathbb{C}^*)$. Then $F(t)$ is a polynomial such that $F(0) = 0$ Moreover it is a multiplicative group homomorphism, e.g. $F(t^2) = F(t)^2$. It follows that $F(t) = t^k$ for some $k \in \mathbb{Z}$. $\square$

A **Laurent monomial** in $n$ variables is defined by

$$t^a = t^{a_1} \cdot t^{a_2} \cdot \ldots \cdot t^{a_n}, \text{ where } a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$$

Observe that $t^a$ defines a function $(\mathbb{C}^*)^n \to \mathbb{C}^*$, i.e. $t^a$ is a character of the torus $(\mathbb{C}^*)^n$. Such character is usually denoted by $\chi^a : T \to \mathbb{C}^*$ where $\chi^a(t) = t^a$.

Another important fact, whose proof can be found in [H] is that:

**Lemma 10.3.7.** *Any irreducible closed subgroup of a torus (i.e. an irreducible affine sub-variety which is a subgroup) is a sub-torus.*

## 10.4 Toric varieties

**Definition 10.4.1.** *A (affine or projective)* **toric variety** *of dimension $n$ is an irreducible (affine or projective) variety $X$ such that*

*(1) $X$ contains an $n$-dimensional torus $T \cong (\mathbb{C}^*)^n$ as Zariski-open subset.*

*(2) the multiplicative action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on $X$.*

*Example* 10.4.2. $\mathbb{C}^n$ is an affine toric variety of dimension $n$.

*Example* 10.4.3. $\mathbb{P}^n$ is a projective toric variety of dimension $n$. The map $(\mathbb{C}^*)^n \to \mathbb{P}^n$ defined as $(t_1, \ldots, t_n) \mapsto (1, t_1, \ldots, t_n)$ identifies the torus $(\mathbb{C}^*)^n$ as a subset of the affine patch $\mathbb{C}^n \subset \mathbb{P}^n$. The action:

$$(t_1, \ldots, t_n) \cdot (x_0, x_1, \ldots, x_n) = (x_0, t_1 x_1, \ldots, t_n x_n)$$

is an extension of the multiplicative action on the torus.

*Example* 10.4.4. Consider the Segre embedding $seg : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by $((x_0, x_1), (y_0, y_1)) \mapsto (x_0 y_0, x_0 y_1, x_1 y_0, x_1 y_1)$. Consider now the map $\phi : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^4$ given by $\phi(t_1, t_2) = (1, t_1, t_2, t_1 t_2)$. Observe that if one identifies $(\mathbb{C}^*)^2$ with the Zariski open $\mathbb{P}^1 \times \mathbb{P}^1 \setminus (V(x_0 - 1) \cup V(y_0 - 1))$ then it is $\phi = seg|_{(\mathbb{C}^*)^2}$. By Lemma 10.3.7 this image is a torus which shows that the torus $(\mathbb{C}^*)^2$ can be identified with a Zariski open of the Segre variety $Im(seg) \subset \mathbb{P}^3$. The torus action of $(\mathbb{C}^*)^2$ on $Im(seg)$ defined by $(t_1, t_2) \cdot (x_0, x_1, x_3, x_a) = (x_0, t_1 x_1, t_2 x_3, t_1 t_2 x_4)$ is by definition an extension of the multiplicative self-action.

## 10.5 Discrete data: polytopes

**Definition 10.5.1.** *A subset $M \subset \mathbb{R}^n$ is called a* **lattice** *if it satisfies one of the following equivalent statements.*

*(1) $M$ is an additive subgroup which is discrete as subset, i.e. there exists a positive real number $\epsilon$ such that for each $y \in M$ the only element $x$ such that $d(x, y) < \epsilon$ is given by $y = x$.*

*(2) There are $\mathbb{R}$-linearly independent vectors $b_1, \ldots, b_n$ such that:*

$$M = \sum_1^n \mathbb{Z} b_i = \{\sum_1^n c_i b_i, c_i \in \mathbb{Z}\}$$

A lattice of rank $n$ is then isomorphic to $\mathbb{Z}^n$.

**Definition 10.5.2.** *Let $A = \{m_1, \ldots, m_d\} \in \mathbb{Z}^n$ be a finite set of lattice points. A combination of the form*

$$\sum a_i m_i, \text{ such that } \sum_a^d a_i = 1, a_i \in \mathbb{Q}_{\geq 0}$$

*is called a* **convex combination**. *The set of all convex combinations of points in $A$ is called the* **convex hull** *of $A$ and is denoted by $Conv(A)$.*

**Definition 10.5.3.** *A* **convex lattice polytope** *$P \subset \mathbb{R}^n$ is the convex hull of a fine subset $A \subset \mathbb{Z}^n$. The dimension of $P$ is the dimension of the smallest affine space containing $P$.*

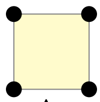
In what follows the term polytope will always mean a lattice convex polytope.

*Example* 10.5.4. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. The polytope $Conv(0, e_1, \ldots, e_n)$ is called the $n$-dimensional regular simplex and it is denoted by $\Delta_n$.

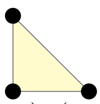
Given a polytope $P = Conv(m_0, m_1, \ldots, m_n)$.

Let $kP = \{m_1 + \ldots + m_k \in \mathbb{R}^n \text{ s.t. } m_i \in P\}$.

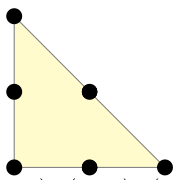
$\Delta_1 = Conv(0, 1)$



$\Delta_1 \times \Delta_1$



$\Delta_2 = Conv((0,0),(0,1),(1,0))$



$2\Delta_2 = Conv((0,0),(0,2),(2,0))$

## 10.6 faces of a polytope

Let $P \subset \mathbb{R}^n$ be an $n$-dimensional lattice polytope. It can be described as the intersection of a finite number of upper-half planes.

**Definition 10.6.1.** *Let $\xi \in \mathbb{Z}^n$ be a vector with integer coordinates and let $b \in \mathbb{Z}$. Define:*

$$H^+_{\xi,b} = \{m \in \mathbb{R}^n \| < m, \xi > \geq b\}, H_{\xi,b} = \{m \in \mathbb{R}^n \| < m, \xi >= b\}$$

$H^+_{\xi,b}$ *is called an* **upper half plane** *and* $H_{\xi,b}$ *is called an* **hyperplane**.

**Definition 10.6.2.** *Let $P \subset \mathbb{R}^n$ be a convex lattice polytope. We say that $H_{\xi,b}$ is a* **supporting hyperplane** *for $P$ if $H_{\xi,b} \cap P \neq \emptyset$ and $P \subset H^+_{\xi,b}$.*

It is immediate to see that a polytope has a finite number of supporting hyperplanes and that:

$$P = \bigcap_{i=1}^{s} H^+_{\xi_i,b_i}$$

**Definition 10.6.3.** *A* **face** *of a polytope $P$ is the intersection of $P$ with a supporting hyperplane. $P$ is considered an (improper) face of itself.*
*Faces are convex lattice polytopes as $Conv(S) \cap H_{\xi,b} = Conv(S \cap H_{\xi,b})$.*
*The dimension of the face is equal to the dimension of the corresponding polytope.*
*Let $F$ be a face, then*

- *$F$ is a* **facet** *if* $\dim(F) = \dim(P) - 1$.
- *$F$ is a* **edge** *if* $\dim(F) = 1$.
- *$F$ is a* **vertex** *if* $\dim(F) = 0$.

*Remark* 10.6.4. Observe (and try to justify) that:

- All polytopes of dimesnsion one are segments.
- All the edges of a polytope contain two vertices.
- $Conv(S)$ contains all the segments between two points in $S$.
- Every convex lattice polytope $P$ is the convex hull of its vertices.

**Definition 10.6.5.** *Let $P, P' \subset \mathbb{R}^n$ be two $n$-dimensional polytopes. They are affinely equivalent if there is a lattice-preserving affine isomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ that maps $P$ to $P'$ and thus biectely $P \cap \mathbb{Z}^n$ to $P' \cap \mathbb{Z}^n$.*

**Definition 10.6.6.** *Let $P$ be a lattice polytope of dimension $n$.*

- *$P$ is said to be simple if through avery vertex there are exactly $n$ vertices.*

- *$P$ is said to be smooth if it is simple and for every vertex $m$ the set of vectors $(v_1 - m, \dots, v_n - m)$, where $v_i$ is the first lattice point on the $i$-th edge, forms a basis for the lattice $\mathbb{Z}^n$.*

*Remark* 10.6.7. All the polygons are simple.

**Lemma 10.6.8.** *a set of vectors $\{v_1, \dots, v_n\} \in \mathbb{R}^n$ is a basis for the lattice $\mathbb{Z}^n$ if and only if the associated matrix $B$ (having the $v_i$ as columns) has determinant* $\pm 1$.

*Proof.* Let $\{v_1, \dots, v_n\} \in \mathbb{R}^n$ is a basis for the lattice $\mathbb{Z}^n$. Then there is an integral matrix $U$ such that $I_n = UB$. Moreover one can observe that the matrix $U$ defines a lattice isomorphism and thus, because the determinant of the inverse has to be an integer, $\det(U) = \pm 1$. $\square$

## 10.7 Assignment: exercises

(1) Consider a minimal hyperplane description of a lattice polytopes $P$. In other words let $P = \bigcap_{i=1}^{s} H^+_{\xi_i, b_i}$ where $s$ is the the minimum number of half-spaces necessary to cut out $P$. Show that $P$ has $s$ facets and that the vectors $\xi_i$ are normal vectors to the associated facet. Moreover show that the pairs $(\xi_i, b_i)$ are uniquely determined up to enumeration ( the vectors $\xi_i$ are unique up to positive scalar factors).

(2) Classify, up to affine equivalence, all the smooth polygons containing at most 8 lattice points.

# Bibliography

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