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Chapter 11

Construction of toric varieties

11.1 Recap example

Example 11.1.1. Consider the Segre embedding $seg : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by $((x_0, x_1), (y_0, y_1)) \mapsto (x_0y_0, x_0y_1, x_1y_0, x_1y_1)$. Consider now the map $\phi : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^4$ given by $\phi(t_1, t_2) = (1, t_1, t_2, t_1t_2)$. Observe that if one identifies $(\mathbb{C})^2$ with the Zariski open $\mathbb{P}^1 \times \mathbb{P}^1 \setminus (V(x_0) \cup V(y_0))$ then it is $\phi = seg|_{(\mathbb{C}^*)^2}$. This image is a torus which shows that the torus $(\mathbb{C}^*)^2$ can be identified with a Zariski open of the Segre variety $Im(seg) \subset \mathbb{P}^3$. The torus action of $(\mathbb{C}^*)^2$ on $Im(seg)$ defined by $(t_1, t_2) \cdot (x_0, x_1, x_3, x_4) = (x_0, t_1x_1, t_2x_3, t_1t_2x_4)$ is by definition an extension of the multiplicative self-action.

Notice that in Example 11.1.1 the map defining the toric embedding and the torus action was given by characters associated to the vertices of the polytope $\Delta_1 \times \Delta_1$. Observe moreover that for this polytope the vertices coincide with all the lattice points in the polytope.

This is of course not always the case, the polytope $2\Delta_2$ for example is the convex hull of 3 vertices, but it contains $|2\Delta_2 \cap \mathbb{Z}^2| = 6$ lattice points.

Example 11.1.2. Let $A = 2\Delta_2 \cap \mathbb{Z}^2 = \{(0,0), (0,1), (1,0), (1,1)(0,2), (2,0)\}$. Consider the map defined by the associated characters and the following composition:

$$\phi_A : (\mathbb{C}^*)^2 \to \mathbb{C}^6 \to \mathbb{P}^5, (t_1, t_2) \mapsto (1, t_1, t_2, t_1t_2, t_1^2, t_2^2).$$

Observe that this map is the restriction of the 2-Veronese embedding. One sees as above that such a variety is a two dimensional projective toric variety.

The previous examples suggest a general construction:

11.2 Toric varieties from polytopes

Let T be an n-dimensional torus with character group $M \cong \mathbb{Z}^n$ and let $A = \{m_0, \ldots, m_d\} \subset M$. Consider the following action of Ton $\mathbb{C}^{d+1}$

$$t \cdot (x_0, \ldots, x_d) = (\chi^{m_0}(t)x_0, \ldots, \chi^{m_d}(t)x_d).$$

This action yields an action on the projective space $\mathbb{P}^d$ as $t \cdot (\lambda x_0, \ldots, \lambda x_d) = \lambda(\chi^{m_0}(t)x_0, \ldots, \chi^{m_d}(t)x_d)$.

Let $x_0 \in \mathbb{P}^d$ be a general points, i.e. a points with non-zero homogeneous coordinates. The orbit $T \cdot x_0 = T_A \cong T$. The Zarisky closure in $\mathbb{P}^d$ of the orbit $x_0 \cong T$ is a projective algebraic variety containing a torus as Zarisky open set.

Let $X_A = \overline{T_A}$ to be such variety.

Alternatively:

Let $P \subset \mathbb{R}^n$ be an n-dimensional polytope and let $A = P \cap \mathbb{Z}^n = \{m_0, \ldots, m_d\}$. Assume that $m_0 = 0$ and that P_A is contained in the positive orthant. Consider the monomial map defined by the associated characters:

$$\phi_A : (\mathbb{C}^*)^n \to \mathbb{C}^{d+1} \to \mathbb{P}^d, (t_1, \ldots, t_n) = t \mapsto (1 : t^{m_1} : \ldots : t^{m_d})$$

The image $Im(\phi_A)$ is a torus T_A. Define X_A to be the Zariski closure of T_A. This means that X_A is the smallest subvariety of $\mathbb{P}^d$ containing T_A. Let $\mathcal{A}$ denote the $n \times (d+1)$ matrix whose columns are the vectors m_i.

Lemma 11.2.1. *Th variety X_A is a projective toric variety of dimension equal to $rank(\mathcal{A})$.*

Proof. Let $T_A = (\mathbb{C}^*)^r$ and consider the lattice of its characters: $Hom_{AG}(T_A, \mathbb{C}^*) = \mathbb{Z}^r$. The map ϕ_A induces a map:

$$Hom_{AG}((\mathbb{C}^*)^{d+1}, \mathbb{C}^*) \to Hom_{AG}((\mathbb{C}^*)^n, \mathbb{C}^*);\ f \mapsto f \circ \phi_A$$

$$\psi_A : \mathbb{Z}^{d+1} \to \mathbb{Z}^n, e_i \mapsto m_i$$

where e_i are the elements of the standard lattice basis. We see that $\psi_A(\mathbb{Z}^{d+1}) = \mathbb{Z}^r$, and thus that $r = rank(\mathcal{A})$. □

Excercise 11.2.2. Consider the $n-$dimensional standard simplex $\Delta_n = Conv(e_0, e_1, \ldots, e_n)$, where $e_0 = 0$. Describe the projective toric variety associated to Δ_n and $2\Delta_n$.

Let $P \subset \mathbb{R}^n$ be an n-dimensional lattice polytope. The toric variety associated to P, denoted by X_P is the topic variety $X_{P\cap\mathbb{Z}^n}$.

11.3 Affine patching and subvarieties

11.4 Recap example

You have seen that $\mathbb{P}^n$ is the projective toric variety associated to the polytope Δ_n. By translating any vertex e_i to $e_0 = 0$ one can contruct a map: $\phi^i : (\mathbb{C}^*)^n \to \mathbb{C}^n \hookrightarrow \mathbb{P}^n$ defined by $t \mapsto (t^{e_0-e_i}, \ldots, 1, t^{e_n-e_i})$. The Zariski closure of $Im(\phi_i)$ defines the affine patch of $\mathbb{P}^n$ where $x_i \neq 0$, i.e.

$$\overline{Im(\phi_i)} = X_i$$

Notice that the map ϕ_i is the map defined by the lattice points:

$$A_i = \{e_0 - e_i, e_i - e_i, \ldots, e_n - e_i\}$$

We will see that projective toric varieties are in a sense a generalisation of projective space as they are built by patching together affine toric varieties defined by the vertices of the polytope.

11.5 Affine patching

Let $P \subset \mathbb{R}^n$ be a polytope and let $A = P \cap \mathbb{Z}^n = \{m_0, \ldots, m_d\}$. For every $m_i \in A$ define $A_{m_i} = \{m - m_i | m \in A\}$. Consider $\phi_{A_m} : (\mathbb{C}^*)^n \to \mathbb{C}^d, t \mapsto (\ldots, t^{m_j - m_i}, \ldots)_{m_j \in A}$ and define:

$$X_m = \overline{Im(\phi_{A_m})} \subset \mathbb{C}^d.$$

Not that X_m is an affine toric variety.

Proposition 11.5.1. *Notation as above. Let $V = \{v_1, \ldots, v_r\}$ be the set of vertices of P. Then*

$$X_A \cong \bigcup_{v_i \in V} X_{v_i}.$$

Proof. First notice that $X_{m_i} = X_A \cap X_i \subset \mathbb{P}^d$ and thus $X_A = \cup_{m \in A} X_m$. We prove the proposition if we show that or every $m \in A$ there is at least one vertex $v \in V$ such that $X_m \subseteq X_v$. As observed $P = Conv(V)$. Let $m = \sum_{v_i \in V} k_i v_i$. After clearing denominators we can write $km = \sum_{v_i \in V} k_i v_i$, for $k_i \in \mathbb{Z}_{\geq 0}$. Notice that $t^m \neq 0$ iff $t^{km} = t^{\sum k_i v_i} = \Pi(t^{v_i})^{k_i} \neq 0$, which happens only if $t^{v_i} \neq 0$ for every $k_i \neq 0$. This shows that $X_m \subseteq X_{v_i}$ for every $k_i \neq 0$. $\square$

The vertices of the polytope defines the affine patches that bild the associated toric variety. The following gives an intuition of how projective toric varieties are considered a generalisation of the projective space.

Excercise 11.5.2. Let P be a polytope of dimension n and let $P \cap \mathbb{Z}^n = \{m_0, \dots, m_d\}$. Show that

- $d \geq n$
- $d = n$ and $m_1, \dots, m_n$ is a lattice basis (i.e. every vector in $\mathbb{Z}^n$ is an integral combination of $m_1, \dots, m_n$) if and only if $P = \Delta_n$.

Let us now examine closer the category of smooth polytopes and the associated topic varieties. Let P be a smooth polytope anklet m_0 be a vertex. After a lattice-preserving affine transformation can we assume that $m_0 = 0$ and that the primitive vectors on the n edges through m_0 are $e_1, \dots, e_n$.

Lemma 11.5.3. *(Exercise) Let P be a smooth polytope. Then $X_v \cong \mathbb{C}^n$ for every vertex v.*

Observe that if P is a n-dimensional smooth lattice polytope, then a facet $F \subset P$ is a smooth polytopes of dimension $(n-1)$. Denote by X_F the associated topic variety.

Lemma 11.5.4. *Let P be a smooth polytope. Then $X_P \setminus T_P = \cup_{F \text{ facet}} X_F$.*

Proof. Let $\dim(P) = n$, let V denote the set of vertices of P and $V(F)$ denote the set of vertices of F. First observe that:

$$X_P \setminus T_P = \cup_{v \in V}(X_v \setminus T_P) = \cup_{v \in V}(\cup_i(\{(x_1, \dots, x_n) \in X_v \text{ s.t. } x_i = 0\})).$$

Let $v = (m_1, \dots, m_n) \in V$, then are n facets passing through $v, F_1, \dots, F_n$ such that $v_i = (m_i, \dots, m_{i-1}, m_{i+1}, m_n) \in V(F_i)$. Clearly it is:

$$\{(x_1, \dots, x_n) \in X_v \text{ s.t. } x_i = 0\} \cong X_{v_i} \subset X_{F_i}.$$

This proves that $X_P \setminus T_P \subseteq \cup_{F \text{ facet}} X_F$. But because for each facet it is $X_F = \cup_{w \in V(F)} X_w$ and $w = v_i$ for some $v \in V$, it is clearly

$$X_F \subset \cup_{v_i = w, w \in V(F)} X_v \setminus T_N \text{ and thus } \cup_{F \text{ facet}} X_F \subseteq X_P \setminus T_P.$$

□

11.6 Assignment: exercises

(1) Prove Lemma 11.5.3

(2) Recall that $kP = \{m_1 + \ldots + m_k \text{ s.t. } m_i \in P\}$ and that if $P_1 \subset \mathbb{R}^n, P_2 \subset \mathbb{R}^t$ then $P_1 \times P_2 = \{(m,n) \text{ s.t. } m \in P_1, m \in P_2\} \subset \mathbb{R}^n \times \mathbb{R}^t$ is a polytope of dimension $\dim(P_1) + \dim(P_2)$ and whose faces are products of faces of resp. polytopes.

  (a) Describe the faces of the polytope $P = \Delta_1 \times 2\Delta_2$.

  (b) Is P smooth?

  (c) Describe the toric variety X_P as union of affine patches.

  (d) Describe the induced map $X_P \to \mathbb{P}^{11}$.