KTH Teknikvetenskap

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11.1 Ideals defined by lattice points

Definition 11.1.1. *A semigroup S is a set with an associative binary operation and an identity* 0.
A semigroup is finitely generated if there is a finite subset $\mathcal{A} \subset S$ such that

$$S = \mathbb{N}\mathcal{A} = \{\sum_{m \in \mathcal{A}} a_m m \text{ s.t. } a_m \in \mathbb{N}\}.$$

Definition 11.1.2. *A finitely generated semigroup $S = \mathbb{N}\mathcal{A}$ is called an affine semigroup if*

- *the binary operation is commutative*
- *It can be embedded in a lattice.*

Let S be an affine semigroup, embedded in the lattice $\mathbb{Z}^n$. We associate to it the so called *semigroup algebra*:

$$\mathbb{C}[S] = \{\sum_{m \in S} c_m \chi^m \text{ s.t. } c_m \in \mathbb{C} \text{ and } c_m = 0 \text{ for all but finitely many } m\}$$

Lemma 11.1.3. *The semigroup algebra $\mathbb{C}[S]$is a subring of the ring of Lurent polynomials in d variables $\mathbb{C}[t_1, t_1^{-1}, ..., t_n, t_n^{-1}]$.*

Proof. The proof is left as exercise. □

Consider an affine toric variety $X_{\mathcal{A}}$, associated to the finite subset $\mathcal{A} \subset \mathbb{Z}^n$. It clearly defines an affine semigroup $S_{\mathcal{A}}$ and a semigroup algebra

$$\mathbb{C}[S_{\mathcal{A}}] = \mathbb{C}[X_{\mathcal{A}}] = \mathbb{C}[\chi^{m_1}, ..., \chi^{m_d}]$$

(associated to the characters of the torus).
Remark 11.1.4. The semigroup algebra associated to the torus $T_{\mathcal{A}}$ is the algebra of all Laurent polynomials in n variables:

$$\mathbb{C}[T_{\mathcal{A}}] = \mathbb{C}[t_1, t_1^{-1}, ..., t_n, t_n^{-1}]$$

Note that $(\mathbb{C}^*)^n \cong V(x_1y_1 - 1, ..., x_ny_n - 1) \subset \mathbb{C}^{2n}$.

Let $A = \{m_0, \dots, m_d\} \subset \mathbb{Z}^n$ as above. Consider the following two maps:

$$\psi_A^* : \mathbb{C}[y_0, \dots, y_d] \to \mathbb{C}[x_1, \cdots, x_n], \psi_A : \mathbb{Z}^{d+1} \to \mathbb{Z}^n$$

defined as:

$$\psi_A^*(y_i) = x^{m_i} \text{ and } \psi_A(e_i) = m_i$$

Let $I_A = Ker(\psi_A^*)$ and $L = Ker(\psi_A)$. Let moreover $I = \{y^\alpha - y^\beta | \alpha, \beta \in \mathbb{N}^d \text{ and } \alpha - \beta \in L\}$.

Lemma 11.1.5. *I_A is a prime ideal of the ring $\mathbb{C}[y_0, \ldots, y_d]$.*

Proof. The kernel of a ring-morphism is always an ideal. Notice that $\mathbb{C}[y_0, \ldots, y_d]/I_A \cong \mathbb{C}[x^{m_0}, \ldots, x^{m_d}]$ and that $\mathbb{C}[x^{m_0}, \ldots, x^{m_d}]$ is an integral domain. $\square$

Proposition 11.1.6.

$$I_A = I.$$

Proof. It is easily checked that $I \subseteq I_A$. Let $\alpha = \sum \alpha_i e_i, \beta = \sum \beta_i e_i \in \mathbb{N}^d$ such that $\alpha - \beta \in L$, i.e. $\sum \alpha_i m_i = \sum_i \beta_i m_i$. Then $t^{\sum m_i \alpha_i} = t^{\sum m_i \beta_i}$ and thus $\psi_A^*(y^\alpha - y^\beta) = 0$. Assume now that $I_A \setminus I \neq \emptyset$ and let $f \in I_A \setminus I$ be the element of minimal (after setting a term order)leading coefficient y^α. After possibly rescaling we can write:

$$f = y^\alpha + f_1 \text{ ,where } f(x^{m_1}, \ldots, x^{m_d}) = 0.$$

It follows that f_1 has a monomial y^β such that $\phi_A^*(y^\alpha) = \phi_A^*(y^\beta)$ and thus $\alpha - \beta \in L$ which implies $y^\alpha - y^\beta \in I$ for $\alpha = \alpha_j e_j, \beta = \beta_j e_j$. It follows that $f_2 = f - (y^\alpha - y^\beta) \in I_A \setminus I$ is an element with lower leading term than f which is impossible. $\square$

11.2 toric ideals

Definition 11.2.1. *A prime ideal $I \subsetneq \mathbb{C}[y_0, \ldots, y_d]$ is called a* **toric ideal** *if it is of the form I_A for some $A \subset \mathbb{Z}^d$.*

Proposition 11.2.2. *(Homogeneous) toric ideals I define toric (projective) varieties and (projective) toric varieties are defined by (homogeneous) toric ideals.*

Proof. Consider a projective toric variety $X_A \subset \mathbb{P}^d$ defined by

$$A = \{m_0, \ldots, m_d\} \subset \mathbb{Z}^n.$$

Let $I \in \mathbb{C}[y_0, \ldots, y_d]$ be the homogeneous ideal defining Y_A. By definition $f(x^{m_0}, \ldots, x^{m_d}) = 0$ for all $f \in I$ which implies $I \subseteq I_A$ and thus $V(I_A) \subseteq X_A$. On the other hand all the polynomials in I_A vanish on $\phi_A((\mathbb{C}^*)^n)$ which implies that $I_A \subseteq I(\phi_A((\mathbb{C}^*)^n))$ and thus $\phi_A((\mathbb{C}^*)^n) \subseteq V(I_A)$. But X_A is the smallest closed subvariety containing $\phi_A((\mathbb{C}^*)^n)$ which implies $X_A = V(I_A)$. $\square$

11.3 Toric maps

Definition 11.3.1. *Let X, Y be toric varieties and let T_X, T_Y be the algebraic tori. A map $f : X \to Y$ is said to be a toric map if*

(1) $f(T_X) \subseteq T_Y$;

(2) $f|_{T_X} : T_X \to T_Y$ is a group homomorphism.

Definition 11.3.2. *A toric map $f : X \to Y$ is equivariant if*

$$f(t \cdot x) = f(t) \cdot f(x).$$

Consider the map $\phi_A : X_A \hookrightarrow \mathbb{P}^d$. This is an equivariant toric map (we call it a toric embedding). In fact $\phi_A(T_X) \subset T_{\mathbb{P}^d}$ and they are related via the following:

$$T_{\mathbb{P}^d} = \mathbb{P}^d \setminus V(x_0 \cdot x_1 \cdot \dots \cdot x_d).$$

$$1 \to \mathbb{C}^* \to (\mathbb{C}^*)^{d+1} \to T_{\mathbb{P}^d} \to 1$$

$$\phi_A : T_{X_A} \to (\mathbb{C}^*)^{d+1} \to T_{\mathbb{P}^d}.$$

Moreover

$$\phi_A(tx) = ((tx)^{m_0}, \dots, (tx)^{m_d}) = \phi_A(t) \cdot \phi_A(x).$$

11.4 Fixed points

Let P be a smooth polytope of dimension n. and and let $V(P)$ denote the set of vertices. For every vertex $v \in V(F)$ there are n facets passing through v, $F_1, \dots, F_n$. Notice that:

$$v = \cap_{i=1}^n F_i$$
$$\cap_1^n V(F_i) = (0, \dots, 0) \in X_v \cong \mathbb{C}^n$$

Every vertex $v \in V(P)$ corresponds to the point $0 \in X_v$ which is the unique point of X_v fixed by the tour action. This means that $|V(P)|$ corresponds to the number of fixed points in X_P.

Example 11.4.1. The torus action on $\mathbb{P}^n$ has $n + 1$ fixed points: $(1 : 0 : \dots : 0), (0 : 1 : \dots : 0), \dots, (0 : \dots : 0 : 1)$.

11.5 Blow up at a fixed point

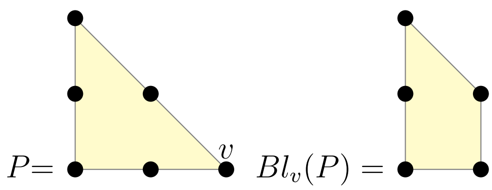
We will define a new polytope, obtained by a give one by truncating a vertex. This is not possible with every polytope and it is for this reason that in this chapter we make the following important assumption.

Definition 11.5.1. *Let P bee a smooth polytope of dimension n. A vertex v is called a* **vertex of order** 2 *if the length of all the n edges through v is at least* 2.

Lat $P = \cap_1^r H^+_{\xi_i,b_i}$ and let v be a vertex of order 2. Let $F_1, \dots, F_n$ be the facets catting v corresponding to $H_{\xi_1,b_1} \cap P, \dots, H_{\xi_n,b_n} \cap P$. We will call the following polytope the **blow up of** P at v and will denote it by $Bl_v(P)$:

$$Bl_v(P) = (\cap_1^r H^+_{\xi_i,b_i}) \cap H^+_{\xi_v,-1}$$

where $\xi_v = \xi_1 + \dots + \xi_n$.



The blow up polytope define a topic variety which will be denoted by $Bl_{x(v)}(X)$ and called the *Blow up of X at the point $x(v)$*. Let $\dim(P) = n$, one can see immediately that:

(1) If $X \subset \mathbb{P}^d$ then $Bl_{x(v)}(X) \subset \mathbb{P}^{d-1}$.

(2) Let $V(P) = \{m_0, \dots, m_d\}$, with $v = m_d$ and let $e_1, \dots, e_n$ be the first integer points on the edges through v. Then $V(Bl_v(P)) = \{m_0, \dots, m_{d-1}, e_1, \dots, e_n\}$.

(3) $H_{\xi_v,-1} \cap Bl_v(P) = Conv(e_1, \dots, e_n) \cong \Delta_{n-1})$

(4) If the facets of P are $H_{\xi_j,b_i} \cap P, i = 1, \dots r$ the the facets of $Bl_v(P)$ are $H_{\xi_j,b_i} \cap Bl_v(P), i = 1, \dots r$ together with $\delta_{n-1} = H_{\xi_v,-1} \cap Bl_v(P)$.

(5) $Bl_v(P)$has the same dimension, n.

Geometrically what happened is that we introduced a $V(\Delta_{n-1}) = \mathbb{P}^{n-1}$ instead of the fixed point $x(v)$.

11.6 Assignment: exercises

(1) A rational normal curve of degree d is defined as the image of the degree d Segre embedding of $\mathbb{P}^1$:

$$\mathbb{P}^1 \to \mathbb{P}^{d+1} \quad (x_0 : x_1) \mapsto (x_0^d : x_0^{d-1}x_1 : x_0^{d-2}x_1^2 : \ldots : x_0 x_1^{d-1} : x_0^d)$$

Let P be a lattice polytope. Show that for every edge $L \subset P$, the toric variety $V(L)$ is smooth and isomorphic to a rational normal curve. What is the degree of such rational curve?

(2) Let $a_0, \ldots, a_n$ be coprime positive integers. Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ given by:

$$t \cdot (x_1, \ldots, x_n) = (t^{a_0}x_0, \ldots, t^{a_n}x_n) = \mathbb{P}(a_0, \ldots, a_n).$$

The quotient $(\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$ exists and it is called the **weighted projective space with weights** $a_0, \ldots, a_n$.

(a) In which sense is this a generalisation of $\mathbb{P}^n$?

(b) We say that a polynomial $p(x) = \sum_\alpha c_\alpha x^\alpha \in \mathbb{C}[x_0, x_1, x_2, \ldots, x_n]$ is $(a_0, a_1, \ldots, a_n)-$homogeneous of weighted degree s if every monomial x^α satisfies $\alpha \cdot (a_0, \ldots, a_n) = s$. Show that $f = 0$ is a well defined equation on $\mathbb{P}(a_0, \ldots, a_n)$ if and only if f is $(a_0, a_1, \ldots, a_n)-$homogeneous.

(c) Consider $\mathbb{P}(1, 1, d)$. Show that the map $\mathbb{P}(1, 1, d) \to \mathbb{P}^{d+1}$ defined by $(x_0, x_1, x_2) \to (x_0^d, x_x^{d-1}x_1, \ldots, x_o x_1^{d-1}, x_1^d, x_2)$ is well defined.

(d) Show that $\mathbb{P}(1, 1, d)$ is a projective toric variety.

(e) Construct the polytope associated to $\mathbb{P}(1, 1, d)$.

(f) (*)[bonus point] Can you show (d) and (e) for any $\mathbb{P}(a_0, \ldots, a_n)$?