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Chapter 10

projective toric varieties and polytopes: definitions

10.1 Introduction

Tori varieties are algebraic varieties related to the study of **sparse polynomials.** A polynomial is said to be sparse if it only contains prescribed monomials.

Let $A = \{m_0, \dots, m_d\} \subset \mathbb{Z}^n$ be a finite subset of integer points. We will use the multi-exponential notation:

$$x^a = x_1^{a_1} \cdot x_2^{a_2} \cdots x_n^{a_n} \text{ where } x = (x_1, \dots, x_n) \text{ and } a = (a_1, \dots, a_n) \in \mathbb{Z}^n$$

Sparse polynomials of type A are polynomials in n variables of type:

$$p(x) = \sum_{a \in A} c_a x^a$$

For example if $A = \{(i, j) \in \mathbb{Z}_+^2 \text{ such that } i + j \leq k\}$ then the polynomials of type A are all possible polynomials of degree up to k.

Toric varieties admit equivalent definitions arising naturally in many mathematical areas such as: Algebraic Geometry, Symplectic Geometry, Combinatorics, Statistics, Theoretical Physics etc.

We will present here an approach coming from Convex geometry and will see that toric varieties represent a natural generalization of projective spaces.

There are two main features we will try to emphasise:

(1) toric varieties, X, are prescribed by **sparse polynomials**, in the sense that they are mapped in projective space via these pre-assigned monomials, whose exponents span an integral polytope **polytope** P_X. You

can think at a parabola parametrized locally by $t \mapsto (t, t^2)$. The monomials are prescribed by the points $1, 2 \in \mathbb{Z}$. The polytope spanned by these points is a segment of length 1, $[1, 2]$. Discrete data A (i.e. points in $\mathbb{Z}^n$) gives rise to a polytope P_A and in turn to a torc variety X_A allowing a geometric analysis of the original data. This turns out to be very useful in Statistics or Bio-analysis for example.

(2) Toric varieties are defined by **binomial ideals**, i.e. ideals generated by polynomials consisting of two monomials: $x^u - x^v$. In the example of the parabola all the points in the image are zeroes of the binomial: $y - x^2$. This feature is particularly useful in integer programming when one wants to find a vertex (of the associated polytope) that minimises a certain (cost) function.

10.2 Recap example

Consider the ideal $(x^3 - y^2) \in \mathbb{C}[x, y]$.

(a) The generating polynomial is irreducible and thus the corresponding affine variety $X = Z(x^3 - y^2) \subset \mathbb{C}^2$ is an **irreducible affine variety**.

(b) Consider now the **algebraic torus** $\mathbb{C}^* = \mathbb{C} \setminus \{0\} \subset \mathbb{C}$. Notice that $\mathbb{C}^* = \mathbb{C} \setminus Z(x)$, a Zariski-open subset of $\mathbb{C}$. Consider now the map $\phi : \mathbb{C}^* \to \mathbb{C}^2$ defined as $\phi(t) = (t^2, t^3)$. Observe that $Im(\phi) \subseteq X$ and that $\psi : Im(\phi) \to \mathbb{C}^*$ defined as $\phi(x, y) = (y/x)$ is an inverse. It follows that $\mathbb{C}^* \cong Im(\phi)$, i.e. $\mathbb{C}^* \subset X$.

(c) The open set $\mathbb{C}^*$ is also a multiplicative group. We can define a **group action** on X as follows:

$$\mathbb{C}^* \times X \to X, (t, (x, y)) \mapsto (t^2 x, t^3 y).$$

Notice that, by definition, the action restricted to $\mathbb{C}^* \subset X$ is the multiplication in the group.

We will call such a variety, i.e. a variety satisfying (a), (b) and (c), an **affine toric variety**

10.3 Algebraic tori

Definition 10.3.1. *A* **linear algebraic group** *is a Zariski-open set G having the structure of a group and such that the multiplication map and the* *inverse map:*

$$m : G \times G \to G, i : G \to G$$

are morphisms of affine varieties.
Let G, G' be two linear algebraic groups, a **morphism** $G \to G'$ *of linear algebraic groups is a map which is a morphism of affine varieties and a homomorphism of groups.*

We will indicate the SET of such morphisms with $Hom_{AG}(G, G')$.
Excercise 10.3.2. Show that when G, G' are abelian $Hom_{AG}(G, G')$ is an abelian group.
Example 10.3.3. The classical examples of algebraic groups are:$(\mathbb{C}^*)^n, GL_n, SL_n$.

Definition 10.3.4. *An n-dimensional algebraic torus is a Zariski-open set T, isomorphic to $(\mathbb{C}^*)^n$.*

An algebraic torus is a group, with the group operation that makes the isomorphism (of affine varieties) a group-homomorphism. Hence an algebraic torus is a linear algebraic group.
From now on we will drop the adjective algebraic in algebraic torus.

Definition 10.3.5. *Let T be a torus.*

- *An element of the abelian group $Hom_{AG}(\mathbb{C}^*, T)$ is called a one parameter subgroup of T.*
- *An element of the abelian group $Hom_{AG}(T, \mathbb{C}^*)$ is called a character of T.*

Lemma 10.3.6. *Let $T \cong (\mathbb{C}^*)^n$ be a torus.*

$$Hom_{AG}(T, \mathbb{C}^*) \cong \mathbb{Z}^n.$$

Proof. Because $Hom_{AG}(T, \mathbb{C}^*) \cong (Hom_{AG}(\mathbb{C}^*, \mathbb{C}^*))^n$ it suffices to prove that $Hom_{AG}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{Z}$. Let $F : \mathbb{C}^* \to \mathbb{C}^*$ be an element of $Hom_{AG}(\mathbb{C}^*, \mathbb{C}^*)$. Then $F(t)$ is a polynomial such that $F(0) = 0$ Moreover it is a multiplicative group homomorphism, e.g. $F(t^2) = F(t)^2$. It follows that $F(t) = t^k$ for some $k \in \mathbb{Z}$. □

A **Laurent monomial** in n variables is defined by

$$t^a = t^{a_1} \cdot t^{a_2} \cdot \ldots \cdot t^{a_n}, \text{ where } a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$$

Observe that t^a defines a function $(\mathbb{C}^*)^n \to \mathbb{C}^*$, i.e. t^a is a character of the torus $(\mathbb{C}^*)^n$. Such character is usually denoted by $\chi^a : T \to \mathbb{C}^*$ where $\chi^a(t) = t^a$.
Another important fact, whose proof can be found in [H] is that:

Lemma 10.3.7. *Any irreducible closed subgroup of a torus (i.e. an irreducible affine sub-variety which is a subgroup) is a sub-torus.*

10.4 Toric varieties

Definition 10.4.1. *A (affine or projective)* **toric variety** *of dimension n is an irreducible (affine or projective) variety X such that*

(1) X contains an n-dimensional torus $T \cong (\mathbb{C}^*)^n$ as Zariski-open subset.

(2) the multiplicative action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on X.

Example 10.4.2. $\mathbb{C}^n$ is an affine toric variety of dimension n.

Example 10.4.3. $\mathbb{P}^n$ is a projective toric variety of dimension n. The map $(\mathbb{C}^*)^n \to \mathbb{P}^n$ defined as $(t_1, \ldots, t_n) \mapsto (1, t_1, \ldots, t_n)$ identifies the torus $(\mathbb{C}^*)^n$ as a subset of the affine patch $\mathbb{C}^n \subset \mathbb{P}^n$. The action:

$$(t_1, \ldots, t_n) \cdot (x_0, x_1, \ldots, x_n) = (x_0, t_1 x_1, \ldots, t_n x_n)$$

is an extension of the multiplicative action on the torus.

Example 10.4.4. Consider the Segre embedding $seg : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by $((x_0, x_1), (y_0, y_1)) \mapsto (x_0 y_0, x_0 y_1, x_1 y_0, x_1 y_1)$. Consider now the map $\phi : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^4$ given by $\phi(t_1, t_2) = (1, t_1, t_2, t_1 t_2)$. Observe that if one identifies $(\mathbb{C}^*)^2$ with the Zariski open $\mathbb{P}^1 \times \mathbb{P}^1 \setminus (V(x_0 - 1) \cup V(y_0 - 1))$ then it is $\phi = seg|_{(\mathbb{C}^*)^2}$. By Lemma 10.3.7 this image is a torus which shows that the torus $(\mathbb{C}^*)^2$ can be identified with a Zariski open of the Segre variety $Im(seg) \subset \mathbb{P}^3$. The torus action of $(\mathbb{C}^*)^2$ on $Im(seg)$ defined by $(t_1, t_2) \cdot (x_0, x_1, x_3, x_a) = (x_0, t_1 x_1, t_2 x_3, t_1 t_2 x_4)$ is by definition an extension of the multiplicative self-action.

10.5 Discrete data: polytopes

Definition 10.5.1. *A subset $M \subset \mathbb{R}^n$ is called a* **lattice** *if it satisfies one of the following equivalent statements.*

(1) M is an additive subgroup which is discrete as subset, i.e. there exists a positive real number ϵ such that for each $y \in M$ the only element x such that $d(x, y) < \epsilon$ is given by $y = x$.

(2) There are $\mathbb{R}$-linearly independent vectors $b_1, \dots, b_n$ such that:

$$M = \sum_1^n \mathbb{Z} b_i = \{\sum_1^n c_i b_i, c_i \in \mathbb{Z}\}$$

A lattice of rank n is then isomorphic to $\mathbb{Z}^n$.

Definition 10.5.2. *Let $A = \{m_1, \dots, m_d\} \in \mathbb{Z}^n$ be a finite set of lattice points. A combination of the form*

$$\sum a_i m_i, \text{ such that } \sum_a^d a_i = 1, a_i \in \mathbb{Q}_{\geq 0}$$

is called a **convex combination***. The set of all convex combinations of points in A is called the* **convex hull** *of A and is denoted by $Conv(A)$.*

Definition 10.5.3. *A* **convex lattice polytope** *$P \subset \mathbb{R}^n$ is the convex hull of a fine subset $A \subset \mathbb{Z}^n$. The dimension of P is the dimension of the smallest affine space containing P.*

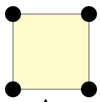
In what follows the term polytope will always mean a lattice convex polytope.

Example 10.5.4. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. The polytope $Conv(0, e_1, \dots, e_n)$ is called the n-dimensional regular simplex and it is denoted by Δ_n.

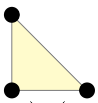
Given a polytope $P = Conv(m_0, m_1, \dots, m_n)$.

Let $kP = \{m_1 + \dots + m_k \in \mathbb{R}^n \text{ s.t. } m_i \in P\}$.

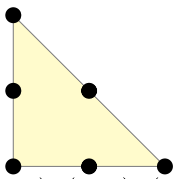
$\Delta_1 = Conv(0, 1)$



$\Delta_1 \times \Delta_1$



$\Delta_2 = Conv((0,0), (0,1), (1,0))$



$2\Delta_2 = Conv((0,0), (0,2), (2,0))$

10.6 faces of a polytope

Let $P \subset \mathbb{R}^n$ be an n-dimensional lattice polytope. It can be described as the intersection of a finite number of upper-half planes.

Definition 10.6.1. *Let* $\xi \in \mathbb{Z}^n$ *be a vector with integer coordinates and let* $b \in \mathbb{Z}$. *Define:*

$$H^+_{\xi,b} = \{m \in \mathbb{R}^n \| < m, \xi > \geq b\}, H_{\xi,b} = \{m \in \mathbb{R}^n \| < m, \xi >= b\}$$

$H^+_{\xi,b}$ *is called an* **upper half plane** *and* $H_{\xi,b}$ *is called an* **hyperplane**.

Definition 10.6.2. *Let* $P \subset \mathbb{R}^n$ *be a convex lattice polytope. We say that* $H_{\xi,b}$ *is a* **supporting hyperplane** *for* P *if* $H_{\xi,b} \cap P \neq \emptyset$ *and* $P \subset H^+_{\xi,b}$.

It is immediate to see that a polytope has a finite number of supporting hyperplanes and that:

$$P = \bigcap_{i=1}^{s} H^+_{\xi_i,b_i}$$

Definition 10.6.3. *A* **face** *of a polytope* P *is the intersection of* P *with a supporting hyperplane.* P *is considered an (improper) face of itself.*
Faces are convex lattice polytopes as $Conv(S) \cap H_{\xi,b} = Conv(S \cap H_{\xi,b})$.
The dimension of the face is equal to the dimension of the corresponding polytope.
Let F *be a face, then*

- F *is a* **facet** *if* $\dim(F) = \dim(P) - 1$.
- F *is a* **edge** *if* $\dim(F) = 1$.
- F *is a* **vertex** *if* $\dim(F) = 0$.

Remark 10.6.4. Observe (and try to justify) that:

- All polytopes of dimesnsion one are segments.
- All the edges of a polytope contain two vertices.
- $Conv(S)$ contains all the segments between two points in S.
- Every convex lattice polytope P is the convex hull of its vertices.

Definition 10.6.5. *Let* $P, P' \subset \mathbb{R}^n$ *be two* n*-dimensional polytopes. They are affinely equivalent if there is a lattice-preserving affine isomorphism* $\phi : \mathbb{R}^n \to \mathbb{R}^n$ *that maps* P *to* P' *and thus biectely* $P \cap \mathbb{Z}^n$ *to* $P' \cap \mathbb{Z}^n$.

Definition 10.6.6. *Let* P *be a lattice polytope of dimension* n.

- P *is said to be simple if through avery vertex there are exactly* n *vertices.*

- *P is said to be smooth if it is simple and for every vertex m the set of vectors $(v_1 - m, \ldots, v_n - m)$, where v_i is the first lattice point on the i-th edge, forms a basis for the lattice $\mathbb{Z}^n$.*

Remark 10.6.7. All the polygons are simple.

Lemma 10.6.8. *a set of vectors $\{v_1, \ldots, v_n\} \in \mathbb{R}^n$ is a basis for the lattice $\mathbb{Z}^n$ if and only if the associated matrix B (having the v_i as columns) has determinant* ± 1.

Proof. Let $\{v_1, \ldots, v_n\} \in \mathbb{R}^n$ is a basis for the lattice $\mathbb{Z}^n$. Then there is an integral matrix U such that $I_n = UB$. Moreover one can observe that the matrix U defines a lattice isomorphism and thus, because the determinant of the inverse has to be an integer, $\det(U) = \pm 1$. □

10.7 Assignment: exercises

(1) Consider a minimal hyperplane description of a lattice polytopes P. In other words let $P = \bigcap_{i=1}^{s} H^{+}_{\xi_i, b_i}$ where s is the the minimum number of half-spaces necessary to cut out P. Show that P has s facets and that the vectors ξ_i are normal vectors to the associated facet. Moreover show that the pairs (ξ_i, b_i) are uniquely determined up to enumeration (the vectors ξ_i are unique up to positive scalar factors).

(2) Classify, up to affine equivalence, all the smooth polygons containing at most 8 lattice points.

Chapter 11

Construction of toric varieties

11.1 Recap example

Example 11.1.1. Consider the Segre embedding $seg : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by $((x_0, x_1), (y_0, y_1)) \mapsto (x_0y_0, x_0y_1, x_1y_0, x_1y_1)$. Consider now the map $\phi : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^4$ given by $\phi(t_1, t_2) = (1, t_1, t_2, t_1t_2)$. Observe that if one identifies $(\mathbb{C})^2$ with the Zariski open $\mathbb{P}^1 \times \mathbb{P}^1 \setminus (V(x_0) \cup V(y_0))$ then it is $\phi = seg|_{(\mathbb{C}^*)^2}$. This image is a torus which shows that the torus $(\mathbb{C}^*)^2$ can be identified with a Zariski open of the Segre variety $Im(seg) \subset \mathbb{P}^3$. The torus action of $(\mathbb{C}^*)^2$ on $Im(seg)$ defined by $(t_1, t_2) \cdot (x_0, x_1, x_3, x_4) = (x_0, t_1x_1, t_2x_3, t_1t_2x_4)$ is by definition an extension of the multiplicative self-action.

Notice that in Example 10.4.4 the map defining the toric embedding and the torus action was given by characters associated to the vertices of the polytope $\Delta_1 \times \Delta_1$. Observe moreover that for this polytope the vertices coincide with all the lattice points in the polytope.

This is of course not always the case, the polytope $2\Delta_2$ for example is the convex hull of 3 vertices, but it contains $|2\Delta_2 \cap \mathbb{Z}^2| = 6$ lattice points.

Example 11.1.2. Let $A = 2\Delta_2 \cap \mathbb{Z}^2 = \{(0,0), (0,1), (1,0), (1,1)(0,2), (2,0)\}$. Consider the map defined by the associated characters and the following composition:

$$\phi_A : (\mathbb{C}^*)^2 \to \mathbb{C}^6 \to \mathbb{P}^5, (t_1, t_2) \mapsto (1, t_1, t_2, t_1t_2, t_1^2, t_2^2).$$

Observe that this map is the restriction of the 2-Veronese embedding. One sees as above that such a variety is a two dimensional projective toric variety.

The previous examples suggest a general construction:

11.2 Toric varieties from polytopes

Let T be an n-dimensional torus with character group $M \cong \mathbb{Z}^n$ and let $A = \{m_0, \dots, m_d\} \subset M$. Consider the following action of Ton $\mathbb{C}^{d+1}$

$$t \cdot (x_0, \dots, x_d) = (\chi^{m_0}(t)x_0, \dots, \chi^{m_d}(t)x_d).$$

This action yields an action on the projective space $\mathbb{P}^d$ as $t \cdot (\lambda x_0, \dots, \lambda x_d) = \lambda(\chi^{m_0}(t)x_0, \dots, \chi^{m_d}(t)x_d)$.

Let $x_0 \in \mathbb{P}^d$ be a general points, i.e. a points with non-zero homogeneous coordinates. The orbit $T \cdot x_0 = T_A \cong T$. The Zarisky closure in $\mathbb{P}^d$ of the orbit $x_0 \cong T$ is a projective algebraic variety containing a torus as Zarisky open set.

Let $X_A = \overline{T_A}$ to be such variety.

Alternatively:

Let $P \subset \mathbb{R}^n$ be an n-dimensional polytope and let $A = P \cap \mathbb{Z}^n = \{m_0, \dots, m_d\}$. Assume that $m_0 = 0$ and that P_A is contained in the positive orthant. Consider the monomial map defined by the associated characters:

$$\phi_A : (\mathbb{C}^*)^n \to \mathbb{C}^{d+1} \to \mathbb{P}^d, (t_1, \dots, t_n) = t \mapsto (1 : t^{m_1} : \dots : t^{m_d})$$

The image $Im(\phi_A)$ is a torus T_A. Define X_A to be the Zariski closure of T_A. This means that X_A is the smallest subvariety of $\mathbb{P}^d$ containing T_A. Let $\mathcal{A}$ denote the $n \times (d+1)$ matrix whose columns are the vectors m_i.

Lemma 11.2.1. *Th variety X_A is a projective toric variety of dimension equal to $rank(\mathcal{A})$.*

Proof. Let $T_A = (\mathbb{C}^*)^r$ and consider the lattice of its characters: $Hom_{AG}(T_A, \mathbb{C}^*) = \mathbb{Z}^r$. The map ϕ_A induces a map:

$$Hom_{AG}((\mathbb{C}^*)^{d+1}, \mathbb{C}^*) \to Hom_{AG}((\mathbb{C}^*)^n, \mathbb{C}^*);\ f \mapsto f \circ \phi_A$$

$$\psi_A : \mathbb{Z}^{d+1} \to \mathbb{Z}^n, e_i \mapsto m_i$$

where e_i are the elements of the standard lattice basis. We see that $\psi_A(\mathbb{Z}^{d+1}) = \mathbb{Z}^r$, and thus that $r = rank(\mathcal{A})$. □

Excercise 11.2.2. Consider the $n-$dimensional standard simplex $\Delta_n = Conv(e_0, e_1, \dots, e_n)$, where $e_0 = 0$. Describe the projective toric variety associated to Δ_n and $2\Delta_n$.

Let $P \subset \mathbb{R}^n$ be an n-dimensional lattice polytope. The toric variety associated to P, denoted by X_P is the topic variety $X_{P \cap \mathbb{Z}^n}$.

11.3 Affine patching and subvarieties

11.4 Recap example

You have seen that $\mathbb{P}^n$ is the projective toric variety associated to the polytope Δ_n. By translating any vertex e_i to $e_0 = 0$ one can contruct a map: $\phi^i : (\mathbb{C}^*)^n \to \mathbb{C}^n \hookrightarrow \mathbb{P}^n$ defined by $t \mapsto (t^{e_0 - e_i}, \dots, 1, t^{e_n - e_i})$. The Zariski closure of $Im(\phi_i)$ defines the affine patch of $\mathbb{P}^n$ where $x_i \neq 0$, i.e.

$$\overline{Im(\phi_i)} = X_i$$

Notice that the map ϕ_i is the map defined by the lattice points:

$$A_i = \{e_0 - e_i, e_i - e_i, \dots, e_n - e_i\}$$

We will see that projective toric varieties are in a sense a generalisation of projective space as they are built by patching together affine toric varieties defined by the vertices of the polytope.

11.5 Affine patching

Let $P \subset \mathbb{R}^n$ be a polytope and let $A = P \cap \mathbb{Z}^n = \{m_0, \dots, m_d\}$. For every $m_i \in A$ define $A_{m_i} = \{m - m_i | m \in A\}$. Consider $\phi_{A_m} : (\mathbb{C}^*)^n \to \mathbb{C}^d, t \mapsto (\dots, t^{m_j - m_i}, \dots)_{m_j \in A}$ and define:

$$X_m = \overline{Im(\phi_{A_m})} \subset \mathbb{C}^d.$$

Not that X_m is an affine toric variety.

Proposition 11.5.1. *Notation as above. Let $V = \{v_1, \dots, v_r\}$ be the set of vertices of P. Then*

$$X_A \cong \bigcup_{v_i \in V} X_{v_i}.$$

Proof. First notice that $X_{m_i} = X_A \cap X_i \subset \mathbb{P}^d$ and thus $X_A = \cup_{m \in A} X_m$. We prove the proposition if we show that or every $m \in A$ there is at least one vertex $v \in V$ such that $X_m \subseteq X_v$. As observed $P = Conv(V)$. Let $m = \sum_{v_i \in V} k_i v_i$. After clearing denominators we can write $km = \sum_{v_i \in V} k_i v_i$, for $k_i \in \mathbb{Z}_{\geq 0}$. Notice that $t^m \neq 0$ iff $t^{km} = t^{\sum k_i v_i} = \Pi(t^{v_i})^{k_i} \neq 0$, which happens only if $t^{v_i} \neq 0$ for every $k_i \neq 0$. This shows that $X_m \subseteq X_{v_i}$ for every $k_i \neq 0$. □

The vertices of the polytope defines the affine patches that bild the associated toric variety. The following gives an intuition of how projective toric varieties are considered a generalisation of the projective space.

Excercise 11.5.2. Let P be a polytope of dimension n and let $P \cap \mathbb{Z}^n = \{m_0, \dots, m_d\}$. Show that

- $d \geq n$
- $d = n$ and $m_1, \dots, m_n$ is a lattice basis (i.e. every vector in $\mathbb{Z}^n$ is an integral combination of $m_1, \dots, m_n$) if and only if $P = \Delta_n$.

Let us now examine closer the category of smooth polytopes and the associated topic varieties. Let P be a smooth polytope anklet m_0 be a vertex. After a lattice-preserving affine transformation can we assume that $m_0 = 0$ and that the primitive vectors on the n edges through m_0 are $e_1, \dots, e_n$.

Lemma 11.5.3. *(Exercise) Let P be a smooth polytope. Then $X_v \cong \mathbb{C}^n$ for every vertex v.*

Observe that if P is a n-dimensional smooth lattice polytope, then a facet $F \subset P$ is a smooth polytopes of dimension $(n-1)$. Denote by X_F the associated topic variety.

Lemma 11.5.4. *Let P be a smooth polytope. Then $X_P \setminus T_P = \cup_{F \; facet} X_F$.*

Proof. Let $\dim(P) = n$, let V denote the set of vertices of P and $V(F)$ denote the set of vertices of F. First observe that:

$$X_P \setminus T_P = \cup_{v \in V}(X_v \setminus T_P) = \cup_{v \in V}(\cup_i(\{(x_1, \dots, x_n) \in X_v \text{ s.t. } x_i = 0\})).$$

Let $v = (m_1, \dots, m_n) \in V$, then are n facets passing through $v, F_1, \dots, F_n$ such that $v_i = (m_i, \dots, m_{i-1}, m_{i+1}, m_n) \in V(F_i)$. Clearly it is:

$$\{(x_1, \dots, x_n) \in X_v \text{ s.t. } x_i = 0\} \cong X_{v_i} \subset X_{F_i}.$$

This proves that $X_P \setminus T_P \subseteq \cup_{F \text{ facet}} X_F$. But because for each facet it is $X_F = \cup_{w \in V(F)} X_w$ and $w = v_i$ for some $v \in V$, it is clearly

$$X_F \subset \cup_{v_i = w, w \in V(F)} X_v \setminus T_N \text{ and thus } \cup_{F \text{ facet}} X_F \subseteq X_P \setminus T_P.$$

□

11.6 Assignment: exercises

(1) Prove Lemma **??**

(2) Recall that $kP = \{m_1 + \ldots + m_k \text{ s.t. } m_i \in P\}$ and that if $P_1 \subset \mathbb{R}^n, P_2 \subset \mathbb{R}^t$ then $P_1 \times P_2 = \{(m, n) \text{ s.t. } m \in P_1, m \in P_2\} \subset \mathbb{R}^n \times \mathbb{R}^t$ is a polytope of dimension $\dim(P_1) + \dim(P_2)$ and whose faces are products of faces of resp. polytopes.

   (a) Describe the faces of the polytope $P = \Delta_1 \times 2\Delta_2$.

   (b) Is P smooth?

   (c) Describe the toric variety X_P as union of affine patches.

   (d) Describe the induced map $X_P \to \mathbb{P}^{11}$.

11.7 Ideals defined by lattice points

Definition 11.7.1. *A semigroup S is a set with an associative binary operation and an identity* 0.
A semigroup is finitely generated if there is a finite subset $\mathcal{A} \subset S$ such that

$$S = \mathbb{N}\mathcal{A} = \{\sum_{m\in\mathcal{A}} a_m m \ s.t.\ a_m \in \mathbb{N}\}.$$

Definition 11.7.2. *A finitely generated semigroup $S = \mathbb{N}\mathcal{A}$ is called an affine semigroup if*

- *the binary operation is commutative*
- *It can be embedded in a lattice.*

Let S be an affine semigroup, embedded in the lattice $\mathbb{Z}^n$. We associate to it the so called *semigroup algebra*:

$$\mathbb{C}[S] = \{\sum_{m\in S} c_m \chi^m \text{ s.t. } c_m \in \mathbb{C} \text{ and } c_m = 0 \text{ for all but finitely many } m\}$$

Lemma 11.7.3. *The semigroup algebra $\mathbb{C}[S]$is a subring of the ring of Lurent polynomials in d variables $\mathbb{C}[t_1, t_1^{-1}, ..., t_n, t_n^{-1}]$.*

Proof. The proof is left as exercise. □

Consider an affine toric variety $X_{\mathcal{A}}$, associated to the finite subset $\mathcal{A} \subset \mathbb{Z}^n$. It clearly defines an affine semigroup $S_{\mathcal{A}}$ and a semigroup algebra

$$\mathbb{C}[S_{\mathcal{A}}] = \mathbb{C}[X_{\mathcal{A}}] = \mathbb{C}[\chi^{m_1}, ..., \chi^{m_d}]$$

(associated to the characters of the torus).
Remark 11.7.4. The semigroup algebra associated to the torus $T_{\mathcal{A}}$ is the algebra of all Laurent polynomials in n variables:

$$\mathbb{C}[T_{\mathcal{A}}] = \mathbb{C}[t_1, t_1^{-1}, ..., t_n, t_n^{-1}]$$

Note that $(\mathbb{C}^*)^n \cong V(x_1y_1 - 1, ..., x_ny_n - 1) \subset \mathbb{C}^{2n}$.

Let $A = \{m_0, \ldots, m_d\} \subset \mathbb{Z}^n$ as above. Consider the following two maps:

$$\psi_A^* : \mathbb{C}[y_0, \ldots, y_d] \to \mathbb{C}[x_1, \cdots, x_n], \psi_A : \mathbb{Z}^{d+1} \to \mathbb{Z}^n$$

defined as:

$$\psi_A^*(y_i) = x^{m_i} \text{ and } \psi_A(e_i) = m_i$$

Let $I_A = Ker(\psi_A^*)$ and $L = Ker(\psi_A)$. Let moreover $I = \{y^\alpha - y^\beta | \alpha, \beta \in \mathbb{N}^d \text{ and } \alpha - \beta \in L\}$.

Lemma 11.7.5. *I_A is a prime ideal of the ring $\mathbb{C}[y_0, \ldots, y_d]$.*

Proof. The kernel of a ring-morphism is always an ideal. Notice that $\mathbb{C}[y_0, \ldots, y_d]/I_A \cong \mathbb{C}[x^{m_0}, \ldots, x^{m_d}]$ and that $\mathbb{C}[x^{m_0}, \ldots, x^{m_d}]$ is an integral domain. □

Proposition 11.7.6.

$$I_A = I.$$

Proof. It is easily checked that $I \subseteq I_A$. Let $\alpha = \sum \alpha_i e_i, \beta = \sum \beta_i e_i \in \mathbb{N}^d$ such that $\alpha - \beta \in L$, i.e. $\sum \alpha_i m_i = \sum_i \beta_i m_i$. Then $t^{\sum m_i \alpha_i} = t^{\sum m_i \beta_i}$ and thus $\psi_A^*(y^\alpha - y^\beta) = 0$. Assume now that $I_A \setminus I \neq \emptyset$ and let $f \in I_A \setminus I$ be the element of minimal (after setting a term order)leading coefficient y^α. After possibly rescaling we can write:

$$f = y^\alpha + f_1 \text{ ,where } f(x^{m_1}, \ldots, x^{m_d}) = 0.$$

It follows that f_1 has a monomial y^β such that $\phi_A^*(y^\alpha) = \phi_A^*(y^\beta)$ and thus $\alpha - \beta \in L$ which implies $y^\alpha - y^\beta \in I$ for $\alpha = \alpha_j e_j, \beta = \beta_j e_j$. It follows that $f_2 = f - (y^\alpha - y^\beta) \in I_A \setminus I$ is an element with lower leading term than f which is impossible. □

11.8 toric ideals

Definition 11.8.1. *A prime ideal $I \subseteq \mathbb{C}[y_0, \ldots, y_d]$ is called a* **toric ideal** *if it is of the form I_A for some $A \subset \mathbb{Z}^d$.*

Proposition 11.8.2. *(Homogeneous) toric ideals I define toric (projective) varieties and (projective) toric varieties are defined by (homogeneous) toric ideals.*

Proof. Consider a projective toric variety $X_A \subset \mathbb{P}^d$ defined by

$$A = \{m_0, \ldots, m_d\} \subset \mathbb{Z}^n.$$

Let $I \in \mathbb{C}[y_0, \ldots, y_d]$ be the homogeneous ideal defining Y_A. By definition $f(x^{m_0}, \ldots, x^{m_d}) = 0$ for all $f \in I$ which implies $I \subseteq I_A$ and thus $V(I_A) \subseteq X_A$. On the other hand all the polynomials in I_A vanish on $\phi_A((\mathbb{C}^*)^n)$ which implies that $I_A \subseteq I(\phi_A((\mathbb{C}^*)^n))$ and thus $\phi_A((\mathbb{C}^*)^n) \subseteq V(I_A)$. But X_A is the smallest closed subvariety containing $\phi_A((\mathbb{C}^*)^n)$ which implies $X_A = V(I_A)$. □

11.9 Toric maps

Definition 11.9.1. *Let X, Y be toric varieties and let T_X, T_Y be the algebraic tori. A map $f : X \to Y$ is said to be a toric map if*

(1) $f(T_X) \subseteq T_Y$;

(2) $f|_{T_X} : T_X \to T_Y$ *is a group homomorphism.*

Definition 11.9.2. *A toric map $f : X \to Y$ is equivariant if*

$$f(t \cdot x) = f(t) \cdot f(x).$$

Consider the map $\phi_A : X_A \hookrightarrow \mathbb{P}^d$. This is an equivariant toric map (we call it a toric embedding). In fact $\phi_A(T_X) \subset T_{\mathbb{P}^d}$ and they are related via the following:

$$T_{\mathbb{P}^d} = \mathbb{P}^d \setminus V(x_0 \cdot x_1 \cdot \dots \cdot x_d).$$

$$1 \to \mathbb{C}^* \to (\mathbb{C}^*)^{d+1} \to T_{\mathbb{P}^d} \to 1$$

$$\phi_A : T_{X_A} \to (\mathbb{C}^*)^{d+1} \to T_{\mathbb{P}^d}.$$

Moreover

$$\phi_A(tx) = ((tx)^{m_0}, \dots, (tx)^{m_d}) = \phi_A(t) \cdot \phi_A(x).$$

11.10 Fixed points

Let P be a smooth polytope of dimension n. and and let $V(P)$ denote the set of vertices. For every vertex $v \in V(F)$ there are n facets passing through v, $F_1, \dots, F_n$. Notice that:

$$v = \cap_{i=1}^n F_i$$
$$\cap_1^n V(F_i) = (0, \dots, 0) \in X_v \cong \mathbb{C}^n$$

Every vertex $v \in V(P)$ corresponds to the point $0 \in X_v$ which is the unique point of X_v fixed by the tour action. This means that $|V(P)|$ corresponds to the number of fixed points in X_P.

Example 11.10.1. The torus action on $\mathbb{P}^n$ has $n+1$ fixed points: $(1 : 0 : \dots : 0), (0 : 1 : \dots : 0), \dots, (0 : \dots : 0 : 1)$.

11.11 Blow up at a fixed point

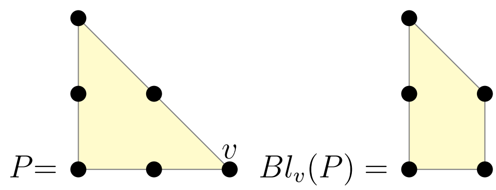
We will define a new polytope, obtained by a give one by truncating a vertex. This is not possible with every polytope and it is for this reason that in this chapter we make the following important assumption.

Definition 11.11.1. *Let P bee a smooth polytope of dimension n. A vertex v is called a* **vertex of order** 2 *if the length of all the n edges through v is at least* 2.

Lat $P = \cap_1^r H^+_{\xi_i,b_i}$ and let v be a vertex of order 2. Let $F_1, \ldots, F_n$ be the facets catting v corresponding to $H_{\xi_1,b_1} \cap P, \ldots, H_{\xi_n,b_n} \cap P$. We will call the following polytope the **blow up of** P at v and will denote it by $Bl_v(P)$:

$$Bl_v(P) = (\cap_1^r H^+_{\xi_i,b_i}) \cap H^+_{\xi_v,-1}$$

where $\xi_v = \xi_1 + \ldots + \xi_n$.



The blow up polytope define a topic variety which will be denoted by $Bl_{x(v)}(X)$ and called the *Blow up of X at the point $x(v)$*. Let $\dim(P) = n$, one can see immediately that:

(1) If $X \subset \mathbb{P}^d$ then $Bl_{x(v)}(X) \subset \mathbb{P}^{d-1}$.

(2) Let $V(P) = \{m_0, \ldots, m_d\}$, with $v = m_d$ and let $e_1, \ldots, e_n$ be the first integer points on the edges through v. Then $V(Bl_v(P)) = \{m_0, \ldots, m_{d-1}, e_1, \ldots, e_n\}$.

(3) $H_{\xi_v,-1} \cap Bl_v(P) = Conv(e_1, \ldots, e_n) \cong \Delta_{n-1})$

(4) If the facets of P are $H_{\xi_j,b_i} \cap P, i = 1, \ldots r$ the the facets of $Bl_v(P)$ are $H_{\xi_j,b_i} \cap Bl_v(P), i = 1, \ldots r$ together with $\delta_{n-1} = H_{\xi_v,-1} \cap Bl_v(P)$.

(5) $Bl_v(P)$has the same dimension, n.

Geometrically what happened is that we introduced a $V(\Delta_{n-1}) = \mathbb{P}^{n-1}$ instead of the fixed point $x(v)$.

11.12 Assignment: exercises

(1) A rational normal curve of degree d is defined as the image of the degree d Segre embedding of $\mathbb{P}^1$:

$$\mathbb{P}^1 \to \mathbb{P}^{d+1} \ \ (x_0 : x_1) \mapsto (x_0^d : x_0^{d-1}x_1 : x_0^{d-2}x_1^2 : \ldots : x_0 x_1^{d-1} : x_0^d)$$

Let P be a lattice polytope. Show that for every edge $L \subset P$, the toric variety $V(L)$ is smooth and isomorphic to a rational normal curve. What is the degree of such rational curve?

(2) Let $a_0, \ldots, a_n$ be coprime positive integers. Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ given by:

$$t \cdot (x_1, \ldots, x_n) = (t^{a_0}x_0, \ldots, t^{a_n}x_n) = \mathbb{P}(a_0, \ldots, a_n).$$

The quotient $(\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$ exists and it is called the **weighted projective space with weights** $a_0, \ldots, a_n$.

(a) In which sense is this a generalisation of $\mathbb{P}^n$?

(b) We say that a polynomial $p(x) = \sum_\alpha c_\alpha x^\alpha \in \mathbb{C}[x_0, x_1, x_2, \ldots, x_n]$ is $(a_0, a_1, \ldots, a_n)-$homogeneous of weighted degree s if every monomial x^α satisfies $\alpha \cdot (a_0, \ldots, a_n) = s$. Show that $f = 0$ is a well defined equation on $\mathbb{P}(a_0, \ldots, a_n)$ if and only if f is $(a_0, a_1, \ldots, a_n)-$homogeneous.

(c) Consider $\mathbb{P}(1, 1, d)$. Show that the map $\mathbb{P}(1, 1, d) \to \mathbb{P}^{d+1}$ defined by $(x_0, x_1, x_2) \to (x_0^d, x_x^{d-1}x_1, \ldots, x_o x_1^{d-1}, x_1^d, x_2)$ is well defined.

(d) Show that $\mathbb{P}(1, 1, d)$ is a projective toric variety.

(e) Construct the polytope associated to $\mathbb{P}(1, 1, d)$.

(f) (*)[bonus point] Can you show (d) and (e) for any $\mathbb{P}(a_0, \ldots, a_n)$?

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