Algebra and geometry through projective spaces

Problems on birational geometry

due: Friday, June 12

Problem 1. Let $f\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map given by $f(x_0 : \dots : x_n) = (y_0 : \dots : y_n)$ where $y_i(x_0, \dots, x_n)$, $i = 0, \dots, n$ are homogeneous polynomials of some common degree d.

(a) Show that f is birational if and only if $y_0 \neq 0$ and the field extension $k(y_1/y_0, \dots, y_n/y_0) \subseteq k(x_1/x_0, \dots, x_n/x_0)$ is an isomorphism.

(b) Give an example of a birational map f as above such that the field extension $k(y_0, y_1, \dots, y_n) \subseteq k(x_0, x_1, \dots, x_n)$ is not an isomorphism.

Problem 2. Let $C \subset \mathbb{P}^2$ be a non-singular conic and let $O \in \mathbb{A}^2 \setminus C$ be a point. Show that there are exactly two tangents of C through O. Conclude that the projection $p\colon C \to \mathbb{P}^1$ from O is two-to-one except in two points P_1 and P_2.

Hint: Use linear coordinate transformations to put C and O into suitable positions.

We say that the map p of the previous problem has *degree* 2, that it is *ramified* in P_1 and P_2, and that it is *branched* in $p(P_1)$ and $p(P_2)$.

Problem 3. Let C be a non-singular cubic curve given on normal form $y^2z = x(x-z)(x-\lambda z)$. Consider the projection from $P = [0:1:0] \in C$. Show that this gives rise to a map $p\colon C \to \mathbb{P}^1$. What is $p(P)$? Determine the degree d of p and its branch locus, i.e., the points $Q \in \mathbb{P}^1$ such that $p^{-1}(Q)$ has fewer than d points.

Problem 4. Consider the Cremona transformation $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by $[x:y:z] \mapsto [yz:zx:xy]$. Prove that this becomes a regular map after blowing up the three points $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ on the source, that is, if $\pi\colon Y \to \mathbb{P}^2$ is the blow-up, then $f \circ \pi$ is defined everywhere. Furthermore, prove that $f \circ \pi$ is isomorphic to the blow-up in the three points $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ on the target.

Hint: $\mathbb{P}^2$ is covered by three standard affine charts. Each blow-up happens in exactly one of these charts giving six affine charts of the blow-up. Put coordinates on these six charts and describe the maps to the source and target of the Cremona map in each of these charts.

Problem 5. Consider the blow-up $\pi\colon \mathrm{Bl}_0\mathbb{A}^3 \to \mathbb{A}^3$ of affine 3-space in the origin $(x,y,z) = (0,0,0)$.

(1) Describe a covering of the blow-up by affine spaces. Determine the equation for the exceptional divisor in each chart.

(2) Give equations for the strict transforms of the hypersurfaces

$$H_1\colon x^2 + y^2 + z^2 = 0,$$
$$H_2\colon x^2 - yz^2 = 0.$$

and determine the multiplicities along the exceptional divisor. Are the strict transforms less singular than the original hypersurfaces?