Homological algebra and algebraic topology
Problem set 11

due: Monday Nov 30 in class.

Problem 1 (3pt). Let $\omega\colon \{0,\dots,n\} \xrightarrow{\cong} \{0,\dots,n\}$ be a permutation of $\{0,\dots,n\}$. In class we associated with w a map $f_w\colon \Delta^n \to \Delta^n$ defined by the formula

$$f_w(t_0,\dots,t_n) = t_0\overline{v}_{\omega(0)} + t_1\frac{\overline{v_{\omega(0)}} + \overline{v_{\omega(1)}}}{2} + \cdots + t_n\frac{\overline{v_{\omega(0)}} + \cdots + \overline{v_{\omega(n)}}}{n+1}$$

For each permutation ω of $\{0,1,2,\dots,n\}$ consider the subspace

$$\Delta^n_\omega = \left\{(t_0,\dots,t_n) \in \Delta^n \mid t_{\omega(0)} \geq t_{\omega(1)} \geq \cdots \geq t_{\omega(n)}\right\} \subset \Delta^n.$$

(1) Prove that Δ^n_ω is the image of f_ω, and f_ω defines a homeomorpism of Δ^n onto its image.
(2) prove that the formula

$$S(\sigma) = \sum_\omega \text{sign}(\omega)\, \sigma \circ f_\omega$$

defines a chain map $S\colon C_*(X) \to C_*(X)$.

Problem 2 (2pt). Let Δ^n be a simplex obtained as the convex hull of $n+1$ points in a general position in $\mathbb{R}^k$, for some k. Let Δ^n_{sd} be a simplex in the barycentric subdivision of Δ^n. Prove that $\text{diam}(\Delta^n_{\text{sd}}) \leq \frac{n}{n+1}\text{diam}(\Delta^n)$. Here diam denotes diameter, with respect to the standard metric on $\mathbb{R}^k$.

Problem 3 (3pt). Prove the following

(1) $\text{H}_0(X,A)$ is always free abelian.
(2) $\text{H}_0(X,A) = \{0\}$ if and only if every path component of X contains a point of A.
(3) $\text{H}_1(X,A) = \{0\}$ if and only if the map $\text{H}_1(A) \to \text{H}_1(X)$ is surjective and every path component of X contains at most one path component of A.

Problem 4 (2pt). Calculate $\text{H}_*(X,A)$ when X is S^2 or $S^1 \times S^1$, and A is a finite subset of X.