INSTITUTIONEN FÖR MATEMATIK, KTH
TENTAMENSKRIVNING, 2005-3-30, KL.10.00-13.00.
5B1473 ELEMENTÄR DIFFERENTIALGEOMETRI

För godkänt behövs att $T + I/7 \geq 14$ (preliminärt), då$T :=$ Tentamenskrivningpoäng och $I :=$ Inlämningsuppgifterpoäng. Inga hjälpmedel.

1) Calculate the curvature (k) and the torsion (τ) of the curve

$$\mathbf{x}(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos t^2 \\ \sin t^2 \\ t^2 \end{pmatrix},$$

preferably by using arc-length parametrisation (3 points)

2) A geodesic is a parametrized curve $c(t) = f(\mathbf{u}(t))$ that ... (write what /as much as/ you know); what happens when reparametrizing, $\mathbf{u}(t) = \tilde{\mathbf{u}}(\tilde{t}(t))$? Find all geodesics on the surface S , which is obtained by rotating $\mathbf{x}(t)$ (as in 1)) around the z-axis; discuss in what sense $\mathbf{x}(t)$ (as in 1)) is /is not/ a geodesic on S. (5 points)

3) Define (what is called) a Gauss-map, and demonstrate its significance by giving some applications. What was Gauss's fundamental discovery concerning isometric (2-dimensional) surfaces? (3 points)

4) Calculate the first and second fundamental form(s), as well as the Christoffel symbols Γ^v_{ab} , and the principal curvatures, of a surface of revolution

$$\mathbf{f}(u, v) = \begin{pmatrix} f(u)\cos v \\ f(u)\sin v \\ u \end{pmatrix};$$

determine all zero-mean-curvature surfaces of this form. (5 points)

5) Derive the Mainardi-Codazzi equations for a hypersurface embedded in $\mathbb{R}^N$, from

$$\partial^2_{ab}\mathbf{f} = h_{ab}\mathbf{n} + \Gamma^c_{ab}\partial_c\mathbf{f}$$

(3 points)

6) What is a manifold? Define the tangent space at a point p of a manifold M, as well as the notion of a tensor field (of type (k,l)) on M. (5 points)

7) Solve ONE of the following three problems (3 points):

A) Show that (and in what way), according to your definition of a manifold (cp.6)), $M = \{[z] | \ z \in \hat{M}\}$ is not /is/ a manifold, if $\hat{M} = \{z = (x, y) | \ -1 < x < 1, y = \pm 1\}$ and $[z]$ denotes the equivalence-class with respect to the equivalence- relation

$$z \sim z' \Leftrightarrow (z = z' \text{ or } x = x' > 0).$$

B) Prove the identity $R^c_{[adb]} = 0$, resp.

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0,$$

satisfied by the curvature tensor of a manifold (you may assume /restrict to/ that X, Y, and Z are tangent vectorfields on a hypersurface embedded in $\mathbb{R}^N$)

C) Calculate explicitely the transformation between two tangent vector-fields, $X^a(x)$ and $\tilde{X}^{\tilde{a}}(\tilde{x})$, on the equator of a 2-sphere, where $\mathbf{x} = (x^1, x^2)$ and $\tilde{\mathbf{x}} = (\tilde{x}^1, \tilde{x}^2)$ are coordinates obtained via stereographic projection(s) from the north- resp. south-pole.