The Poincaré Half-Plane $\mathbb{P}_+ = (\mathbb{M}, g)$

$\mathbb{M} := \{(u,v) \in \mathbb{R}^2 \mid v > 0\}$, $(g_{ab}) := \frac{1}{v^2}\begin{pmatrix}1&0\\0&1\end{pmatrix}$

19) Find all geodesics on $\mathbb{P}_+$
(Calculate the Christoffelsymbols $\Gamma^{\cdot}_{\cdot\cdot}$;
integrate $\ddot{u} + \Gamma^u_{ab}\dot{u}^a\dot{u}^b \overset{(1)}{=} 0$;
notice that $\ddot{v} + \Gamma^v_{ab}\dot{u}^a\dot{u}^b \overset{(2)}{=} 0$, together with (1), implies $g_{ab}\dot{u}^a\dot{u}^b = \frac{1}{v^2}(\dot{u}^2+\dot{v}^2) \overset{(3)}{=} \text{const}$;
integrate (3) by using the result of (1))

20) Show that for an ordinary 2-dimensional surface embedded in $\mathbb{R}^3$ the Gauß-curvature K is $\underset{(4)}{=} -\frac{1}{2}e^{-\varphi(u,v)}(\varphi_{uu}+\varphi_{vv})$, if $(g_{ab}) = e^{\varphi}\begin{pmatrix}1&0\\0&1\end{pmatrix}$;
evaluate the r.h.s. of (4) for the metric in $\mathbb{P}_+$

21) Show that $\varphi: w := u+iv \to \frac{aw+b}{cw+d} =: z = x+iy$ $\left(\begin{pmatrix}a&b\\c&d\end{pmatrix} \in SL(2,\mathbb{R})\right)$
maps $\mathbb{M}$ onto $\mathbb{M}$ 'isometrically', i.e.
$g_p(X,Y) = g_{\varphi(p)}(d\varphi|_p X, d\varphi|_p Y)$, resp.
$g_{ab}du^a du^b = \frac{du^2+dv^2}{v^2} = \frac{dw\,d\bar{w}}{(\operatorname{Im} w)^2} \overset{!}{=} \frac{dz\,d\bar{z}}{(\operatorname{Im} z)^2} = \frac{dx^2+dy^2}{y^2}$