Selected Problems, Set # 5
(Some "topological" analysis)

(1) *Definition.* An n-dimensional vector space consisting of continuous functions on some compact metric space is a *Haar system* (of dimension n) if every non-null function in the collection vanishes at no more than $n-1$ distinct points. (Example: the set of all polynomials of degree at most $n-1$, on $[0,1]$).

Prove: A (real) Haar system on the set $\{(x,y) : x^2+y^2=1\}$ must have *odd* dimension; and, for every odd n, such a system exists. ["real" means, real valued functions and real scalars]

(2) Let $K \subset \mathbb{R}^2$ be that compact set consisting of

$$\{(x,y) : -1 \leq x \leq 1, y=0\} \cup \{(x,y) : x=0, 0 \leq y \leq 1\}.$$

Prove there is no (real) 2-dimensional Haar system on K.

(3) Let K be the set of complex numbers z such that $|z| \leq 1$, and suppose S is a (complex!) 2-dimensional Haar space on K. *Prove:* S contains a function which vanishes nowhere on K.

* Is the analogous assertion for 3-dimensional Haar spaces true?

Prove also: a *real* Haar system on $[0,1]$ contains a non-negative function.

(4) Let A denote any vector subspace of real Euclidean n-space (represented as n-tuples), and $A^\perp$ its orthogonal complement. *Prove* at least one of the spaces A, $A^\perp$ contains a unit vector all of whose coordinates are non-negative.

(5) Let Γ be a Jordan curve, α a point inside.

Prove: There exist points z_1, z_2 on Γ whose midpoint is α. Can we moreover choose z_1, z_2 so the segment joining them has no other intersection with Γ?

(6) Let f be a function from $\mathbb{C}$ (the complex plane) to $\mathbb{C}$ such that, whenever $|z_1 - z_2| = 1$ we have $|f(z_1) - f(z_2)| = 1$.

Prove: $f(z) = az + b$, or else $f(z) = a\bar{z} + b$ (a, b complex constants).

(7) Let E and F be countably infinite sets in $\mathbb{R}^2$, neither of which has a finite limit point. *Prove* their complements are homeomorphic.

(8) Let K be a compact connected set in the plane, and P, Q distinct points of K. *Prove* for every positive integer n there exists a pair of points P', Q' in K such that $P'Q'$ is parallel to PQ, and has length $\frac{1}{n} \cdot (\text{length } \overline{PQ})$.

(9) Prove, in # 8, that the analogous assertion with $\frac{1}{n}$ changed to $\frac{2}{5}$ (or, in fact, to any positive number that is not the reciprocal of an integer) is *false*.

(10) Let T be a torus, on which a closed curve Γ has been drawn with no self intersections. Let m be the number of times Γ winds around the "ring"., and n the number of times it winds around the "hole". *Prove* the greatest common divisor of m, n is 1; and conversely, given m, n with greatest common divisor =1, there exists a curve Γ with the stated properties.

(11) *Prove* one cannot have, in the plane, uncountably many disjoint sets each homeomorphic to the letter T. (*Afterthought:* this may be too hard! You may assume each "T" is made of 3 *segments*:

(12) Can the open interval $(0, 1)$ be decomposed into a union of disjoint *closed* (non-degenerate) intervals?

(13) *Prove* the function

$$f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

cannot be represented in the form $f(x) = \lim_{n\to\infty} f_n(x)\ \forall x \in \mathbb{R}$ where each f_n is continuous.

(14) Let H denote the usual Hilbert space of real square-summable sequences. *Prove* there is no continuous map $\mathbb{R} \to H$ whose range is all of H. Can the range of such a map be dense in H?