PROBLEM SOLVING SEMINAR, SPRING 1974.

List # 1, **Topology and set theory.**

(1) Let Γ be a Jordan curve, α a point inside.

Prove: There are points z_1, z_2 on Γ whose midpoint is α.

(2) Let E, F be countably infinite sets in $\mathbb{R}^2$, neither of which has a finite limit point. Prove their complements are homeomorphic.

(3) Prove one cannot have uncountably many disjoint *triodes* in the plane. (A triode means here the set formed by three closed nondegenerate line segments, having one common endpoint and different directions, like the letter Y. The lengths and directions of the segments may vary arbitrarily from one triode to the next. If you solve the problem, you could try the somewhat harder variant whereby the line segments are replaced by arbitrary Jordan arcs.)

(4) Prove that the function

$$f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

cannot be represented in the form $f(x) = \lim_{n\to\infty} f_n(x) \quad \forall x \in \mathbb{R}$, where each f_n is continuous.

(5) Let $f(t)$ be the even, continuous function of period 2 defined in the interval $[0,1]$ as follows: $f(t) = 0$ in $[0,1/3]$, $f(t) = 1$ in $[2/3,1]$ and f is linear in between. Consider the mapping from $[0,1]$ into the square $(0 \le x \le 1,\ 0 \le x \le 1) \subset \mathbb{R}^2$ given by

$$x = x(t) = \frac{1}{2}f(t) + \frac{1}{2^2}f\left(3^2t\right) + \frac{1}{2^3}f\left(3^4t\right) + \dots,$$
$$y = y(t) = \frac{1}{2}f(3t) + \frac{1}{2^2}f\left(3^3t\right) = \frac{1}{2^3}f\left(3^5t\right) + \dots.$$

Prove the range of this map is the full square ("Peano curve").

(6) Prove that a rectifiable Jordan arc in $\mathbb{R}^2$ has planar Lebesgue measure Q.

(7) Let H denote the standard separable real Hilbert space of square summable functions on $[0,1]$. Prove there is no continuous map $\mathbb{R} \to H$ whose range is all of H, but that this range can be dense.

(8) Let E be a subset of $\mathbb{R}^n$ of positive measure, and $E - E$ the set of all differences of pairs of elements of E. Prove that $E - E$ contains a neighbourhood of the origin.

(9) Let $E \subset \mathbb{R}$ have positive measure. Prove E contains 3 points x_1, x_2, x_3 with $x_1 + x_3 = 2x_2$. Can you generalize this?

(10) Let $f(x)$ be an arbitrary (not necessarily measurable) complex-valued bounded function on $[0,1]$ and consider the totality of all *Riemann sums* formed from f, that is sums of the form

$$S = \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1});$$
$$x_0 = 0 < x_1 < \cdots < x_n = 1,\ x_{j-1} \le \xi_j \le x_j.$$

Let R be the "Riemann set" of f, that is the set of all numbers $\sigma = \lim\limits_{n\to\infty} S_n$ whereby it is assumed that the maximal subinterval in the partition of $[0,1]$ corresponding to S_n tends to 0. Prove R is convex.

(11) Let f be an arbitrary real-valued function of $\mathbb{R}$. Prove: The set of points where f possesses both a right and a left hand derivative, which are not equal to each other, is at most countable.

(12) Suppose infinitely many pairwise disjoint open circular discs are removed from the open unit disc such that the residual set has planar Lebesgue measure 0. Prove that the radii r_n of these discs satisfy $\sum\limits_{n=1}^{\infty} r_n = \infty$.

(13) Let E be the set of real numbers in $(0,1)$ with nonterminating decimal expansions. Let S be a measurable subset of E such that if $x \in S$ every number whose decimal expansion agrees with that of x except in at most finitely many places also belongs to S. Prove the measure of S is either 0 or 1.

(14) Prove: a perfect subset of $[0,1]$ (i.e. a closet subset, each point of which is a cluster point of the set) has the same cardinality as the set of all real numbers.

(15) Can the open interval $(0,1)$ be decomposed into a union of disjoint closed (non-degenerate) intervals?

(16) Let $f(x)$ be a real valued function on $[0,1]$, continuous and of bounded variation. For $t \in \mathbb{R}$ let $n(t)$ denote the (possibly infinite) number of points x where $f(x) = t$. Prove that $\{t : n(t) = \infty\}$ has measure zero.

(17) Let E be a measurable subset of $\mathbb{R}$ with measure $|E| > 0$ and $E + a$ the set obtained upon adding the number a to each element of E. Prove that for all sufficiently small positive a the intersection of $E + a$ with E has measure not exceeding $|E| - a$.

(18) Let Γ be any Jordan arc in $\mathbb{C}$ joining the points $z = 0$ and $z = 1$. Prove there exist points z_1, z_2 on Γ with $z_2 - z_1 = \frac{1}{5}$, but there need not exist any pair with $z_2 - z_1 = \frac{2}{5}$.

(19) Suppose a thread is wound tightly round a torus without crossing itself, and the ends join to form a closed curve. Suppose the thread winds m times around the "ring" and n times around the "hole". Prove: the greatest common divisor of m and n is 1, and for every pair m, n with greatest common divisor $= 1$ there exists a winding yielding these values.

(20) Prove that $\mathbb{R}^2$ is not a union of mutually disjoint circles (= circumferences). (Can you do the harder problem with "circle" replaced by "Jordan curve"? Can you show that $\mathbb{R}^3$ *is* the union of mutually disjoint curves, each homeomorphic to a circle?)