List # 2, **Geometric and combinatorial problems.**

(1) Consider a convex body (= compact convex set) in $\mathbb{R}^n$. A *diameter* of the body is a longest chord in its direction. *Prove* an interior point of a convex body, every chord through which is a diameter, is a center of symmetry of the body.

(2) Let K be a planar convex body. Prove its boundary contains four points which are vertices of a square.

(3) Let K be a planar convex body such that, no matter how it is oriented in the plane it covers at least one lattice point. Prove there is a subset of K which is a square of side 1.

(4) Prove that a convex body in $\mathbb{R}^2$ cannot be partitioned into seven subsets of equal area by three non-concurrent lines.

(5) (Continuation) If 6 of the 7 subsets have equal area, these must be the 6 outer ones.

(6) Prove: every chord through the centroid (= center of gravity) of a planar convex body is divided into two segments, the smaller of which is at least half the larger. And if the smaller is ever exactly half the larger, the body is a triangle.

(7) Let S be a k-dimensional subspace of $\mathbb{R}^n$. The number of open orthants intersected by S is at most

$$2\sum_{i=0}^{k-1}\binom{n-1}{i}.$$

(Note: the term "orthants" refers to the 2^n regions of $\mathbb{R}^n$ determined by the signs of the coordinates; for $n = 2$ they are called quadrants, for $n = 3$ octants.)

(8) If S is a subspace of $\mathbb{R}^n$ and $S^\perp$ its orthocomplement, prove $S \cup S^\perp$ contains a nonzero vector all of whose coordinates are non-negative.

(9) Given $2n$ people, each one a friend of at least n others, prove there is a way of seating them round a table so that only friends sit next to one another.

(10) Let X be a separable metric space such that every set of $n+1$ ($n = 2, 3, 4, \ldots$) distinct points of X can be embedded isometrically in $\mathbb{R}^n$. Prove X can be embedded isometrically in Hilbert space (i.e., there is a distance-preserving map $X \to l^2$).

(11) Consider the real line as a metric space with distance function $d(x_1, x_2) = |x_1 - x_2|^{1/2}$. Prove this space can be embedded isometrically in Hilbert space.

(12) Given $n^2 + 1$ distinct positive integers written in a row, prove there are $n+1$ of them which, in the order written, are a monotone sequence (increasing or decreasing).

(13) Given a finite collection of closed balls (of arbitrary radii) in $\mathbb{R}^n$ which have a common point, suppose we displace them so that no pair of centers is further separated than before. Prove the displaced balls again have a common point.

(14) Given a compact set $K \subset \mathbb{R}^2$ and a map $f : K \to \mathbb{R}^2$ which is contractive (i.e., no pair of points is moved further apart by f). Prove f can be extended to a contractive mapping $\mathbb{R}^2 \to \mathbb{R}^2$.

(15) Let z_j be a complex number of modulus 1, $j = 1, 2, \ldots, 2n+1$. Show that for suitably chosen $\pm$ signs

$$|z_1 \pm z_2 \pm z_3 \pm \cdots \pm z_{2n+1}| \leq 1.$$

(16) Show that any arc of length 1 can be covered by a closed rectangle of area $1/4$.

(17) If one looks at more than half the subsets on n objects, show that at least one pair of these sets is mutually disjoint.

(18) Given an odd number of disjoint points of $\mathbb{R}^n$, prove that there is a *unique* point in $\mathbb{R}^n$ the sum of whose distances from these points is a minimum. Show also that for $n \geq 2$ and an *even* number of points, provided these are not collinear, the same conclusion holds.

(19) Let f be any function from $\mathbb{C}$ to $\mathbb{C}$ such that $|f(z_1) - f(z_2)| = 1$ whenever $|z_1 - z_2| = 1$. Prove that either $f(z) = az + b$ or $f(z) = a\bar{z} + b$ where a, b are complex numbers, $|a| = 1$.

(20) Prove that complex multiplication can be accomplished with 3 real multiplications, not with 2, however. (Observe that the "naive" method is to compute

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

requires 4 real multiplications.)

(21) Suppose you are given, in the plane, a parallelogram, a line λ, and point P on λ. Show how, using only a ruler, a line can be drawn through P parallel to λ. (The ruler is to be used in the classical way, i.e. no markings may be made upon it).

(22) Suppose you are given, in the plane, a parallelogram, a line λ, and a point P on λ. Is it possible, using only a ruler, to draw a line through P perpendicular to λ?

(23) Suppose you are given, in the plane, a line λ and a point P on λ. You have a ruler, on the edge of which two distinguishable points have been marked. Is it possible to draw a perpendicular to λ through the point P?