Problem73# 2.4 Solution

Hölder's inequality:

$$\sum_i^n a_i b_i \le \left(\sum a_i^{1/\theta}\right)^\theta \cdot \left(\sum b_i^{1/(1-\theta)}\right)^{1-\theta}$$

can be shown by means of Young's inequality and a scaling argument.

Young's inequality in its simplest form states:
$xy \le \theta x^{1/\theta} + (1-\theta) y^{1/(1-\theta)} \qquad (0 < \theta < 1,\ x, y > 0).$

To prove Hölder's inequality, which is first order homogeneous in both a_i and b_i, we can assume that $\sum_i^n a_i^{1/\theta} = \sum_i^n b_i^{(1/(1-\theta)} = 1$.
Then from Young and the assumptions:

$$\sum_i^n a_i b_i \le \sum_i^n \theta a_i^{1/\theta} \ + \ \sum_i^n (1-\theta) \sum_i^n b_i^{1/(1-\theta)} = \theta + (1-\theta) = 1 =$$
$$= \left(\sum_i^n a_i^{1/\theta}\right)^\theta \cdot \left(\sum_i^n b_i^{1/(1-\theta)}\right)^{1-\theta}.$$

To prove Young's inequality (which is the convexity part of the proof) we use the convexity of the curve $y = \exp(x)$, which implies that every chord lies above the curve.
Let $t > s$ and consider the chord through the points $(s, \exp(s))$ and $(t, \exp(t))$ on the exponential curve.
If $0 < \theta < 1$ the point $(\theta s + (1-\theta)t,\ \ \theta \exp(s) + (1-\theta)\exp(t))$ belongs to the chord. In view of the convexity of the exponential curve we get:

$$\exp(\theta s + (1-\theta)t) \le \theta \exp(s) + (1-\theta)\exp(t)$$

Setting $x = \exp(s\theta)$ and $y = \exp(t(1-\theta))$ (i.e. $\exp(s) = x^{1/\theta}$, $\exp(t) = y^{1/(1-\theta)}$) we get the required inequality: $xy \le \theta x^{1/\theta} + (1-\theta) y^{1/(1-\theta)}$.

The above proof can of course also be done by means of the concavity of the logarithmic curve.

GJ