Problem 73# 2.5 Solution B

Case n=3:
Set $f(x_1,x_2,x_3) = \dfrac{x_1}{x_2+x_3} + \dfrac{x_2}{x_3+x_1} + \dfrac{x_3}{x_1+x_2}$ and show that $\inf(f) \geq \frac{3}{2}$ in $x_1,x_2,x_3 > 0$.
Suppose wlog that $x_1+x_2+x_3 = 1$ and study the the behaviour of f on the simplex $\Omega : \sum x_j = 1,\ x_j > 0, j = 1.2.3.$ and its border $\partial\Omega$.

On Ω :
Set $g(x_1,x_2) = f(x_1,x_2,1-x_1-x_2) = \dfrac{x_1}{1-x_1} + \dfrac{x_2}{1-x_2} + \dfrac{1-x_1-x_2}{x_1+x_2} = 1 - \dfrac{x_1}{1-x_1} - \dfrac{x_2}{1-x_2} + \dfrac{1}{x_1+x_2}$

Then

$$\frac{\partial g}{\partial x_1} = \frac{1}{(1-x_1)^2} - \frac{1}{(x_1+x_2)^2} = 0$$
$$\frac{\partial g}{\partial x_2} = \frac{1}{(1-x_2)^2} - \frac{1}{(x_1+x_2)^2} = 0$$

immediately yields the unique stationary point $(x_1,x_2,x_3) = (1/3,1/3,1/3)$ on Ω which gives the value $f(1/3,1/3,1/3) = 3/2$.

On $\partial\Omega$:
The value of f tends to $+\infty$ when the corner points $(1,0,0),(0,1,0),(0,0,1)$ are approached from within Ω. On the rest of $\partial\Omega$ f is defined. On e.g. $x_1+x_2 = 1, x_3 = 0,\quad f = \dfrac{x_1}{x_2} + \dfrac{x_2}{x_1}$ which has the minimum 2 on $\partial\Omega$ and similarly on $x_2+x_3 = 1, x_1 = 0$ and $x_3+x_1 = 1, x_2 = 0$.

Hence, in case $n = 3$, the infimum $3/2$ is an attained minimum at the point $(1/3,1/3,1/3)$.