Problem 73 # 2.5 Solution D

We shall show (following Mordell) that for n equal to $3, 4, 5$ or 6 we have

$$\sum_{j=1}^{n} x_j/B_j \geq n/2$$

where x_j are non-negative real numbers such that $B_j := x_{j+1} + x_{j+2}$ are all positive (here subscripts larger than n are to be diminished by n). Moreover, equality holds only in the cases where all the B_j are equal, which occurs precisely when either all the x_j are equal, or n is even and
$x_1 = x_3, x_2 = x_4, ...x_{n-2} = x_n$.

Proof. By Cauchy's inequality

$$\left(\sum_{j=1}^{n} x_j\right)^2 \leq \sum_{j=1}^{n} x_j/B_j \cdot \sum_{j=1}^{n} x_j B_j$$
.

Apart from identifying the cases of equality, the inequality to be proved follows if we can show
$$\sum_{j=1}^{n} x_j B_j \leq (2/n)\left(\sum_{j=1}^{n} x_j\right)^2.$$

We consider cases. For $n = 3$ we have to show

$3[x_1(x_2 + x_3) + x_2(x_3 + x_1) + x_3(x_1 + x_2)]$

$\leq 2(x_1 + x_2 + x_3)^2$, or simplifying:

$x_1x_2 + x_2x_3 + x_3x_1 \leq (x_1)^2 + (x_2)^2 + (x_3)^2$

and the last inequality is a consequence of Cauchy's inequality.

Thus, the case $n = 3$ is proved, and the equality is seen to be possible only when it is so for each of the two applications of Cauchy's inequality; thus we must have proportionality of the vectors

(B_1, B_2, B_3) and $(1/B_1, 1/B_2, 1/B_3)$

as well as that of the pair

(x_1, x_2, x_3) and (x_2, x_3, x_1)

.

In particular, there is a positive t such that $x_2 = tx_1, x_3 = tx_2$ and $x_1 = tx_3$ which implies $t = 1$ so all the x_j are equal.

The case $n = 4$: Here we have to show

$$2\sum_{j=1}^{4} x_j B_j \leq \left(\sum_{j=1}^{4} x_j\right)^2$$

or

$$2\big(x_1(x_2+x_3)+x_2(x_3+x_4)+x_3(x_4+x_1)+x_4(x_1+x_2)\big) \leq (x_1+x_2+x_3+x_4)^2,$$

or, simplifying:

$$2(x_1x_3 + x_2x_4) \leq \sum_{j=1}^{4} x_j^2$$

The latter is again a consequence of Cauchy's inequality, so the case $n = 4$ is settled, apart from the cases of equality.
The first use of Cauchy's inequality becomes equality only if the vectors

(B_1, B_2, B_3, B_4) and $(1/B_1, 1/B_2, 1/B_3, 1/B_4)$ are proportional which implies $B_1 = B_2 = B_3 = B_4$, i.e.

(*) $x_1 = x_3 \quad$ and $\quad x_2 = x_4$.

Moreover these conditions are precisely those which give equality in the second application of Cauchy's inequality. Thus, (*) furnish the necessary and sufficient conditions for equality in the cyclic inequality when $n = 4$.

The case $n = 5$ is like that of n=3, and $n = 6$ like that of $n = 4$ (although much more computation is needed) and we leave the details to the reader.