Problem73# 2.6 Solution

Let us denote by a the vector $(a_1, \ldots, a_n)$ and similarly for b. Also, let

$$G(a) = G(a_1, \ldots, a_n) := (a_1 a_2 \ldots a_n)^{(1/n)}$$

denote the geometric mean of the a_i, and

$$A(a) = A(a_1, \ldots, a_n) := (a_1 + \cdots + a_n)/n$$

the arithmetic mean (and likewise for $G(b)$ and $A(b)$. In this notation the statement to be proved can be written:

$$G(a + b) \ \geq G(a) + G(b), \tag{1}$$

where $a + b$ is the usual vector sum of a and b.

Now, $G(a)/G(a + b) = G(a_1/(a_1 + b_1), \ldots, a_n/(a_n + b_n)) \leq A(a_1/(a_1 + b_1), \ldots, a_n/(a_n + b_n))$, and we get in the same way that $G(b)/G(a + b) \leq A(b_1/(a_1 + b_1), \ldots, b_n/(a_n + b_n))$.

Adding these, and using the linearity of $A(\cdot)$ as a function of its vector argument gives

$$[G(a) + G(b)]/G(a + b) \ \leq A(1, 1, ..., 1) = 1,$$

which is equivalent to the required inequality. To have equality (we'll only discuss the case where none of the a_i nor b_i is 0) we must have $a_i/(a_i + b_i) = a_j/(a_j + b_j)$ for all i, j which implies that $b = ta$ for a positive scalar t. We leave to the reader to complete the discussion of cases of inequality.

HSS