Potential theory

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1 Background

Potential theory has its physical origin in Newton's and Coulomb's laws of gravitational and electrostatic attraction. These laws of distant action invite for concepts of fields which mediate the action. The fields themselves are vector fields which satisfy rather involved systems of equations (like Maxwell's equations), but part of these equations can be automatically resolved by setting the field up as the gradient a scalar *potential* field, and then the remaining part become a quite tractable equation for this potential, namely Laplace's equation.

Thus classical potential theory studies solutions of Laplace's equation, namely *harmonic* functions, and more generally sub- and superharmonic functions. In the case of two dimensions, the field equations reduce to essentially the Cauchy-Riemann system, the solutions of which are the analytic functions. Mathematical potential theory originated in the work of Green, Gauss and others in the early 19th century, and has since that time served as a powerful tool both in treating equations arising in physics and in the pure mathematical theory of analytic functions.

During the 20th century mathematical potential theory developed in many directions (e.g., parabolic, probabilistic, abstract and discrete potential theory, pluripotential theory and various kinds of nonlinear potential theory). At the same time new kinds of potentials came up in physics: in Einstein's theory of general relativity (which has Newton's theory as a limiting case) the role of potentials is played by the coefficients of the four dimensional space-time metric, and in gauge field theories, for the fundamental forces besides gravitation, the role of potentials is taken by connection coefficients. The famous Aharonov-Bohm effect shows that in such contexts the potential has a real physical meaning, and is not just a mathematical tool (as in classical physics).

Two dimensional potential theory is particularly rich because of its connections to analytic function theory, and the step from two dimensions to higher dimensions is very easy. Thus the two-dimensional theory is a suitable starting point. A good text book here is [2].

2 Problems to think of

1) Let S^2 be the unit sphere in $\mathbf{R}^3$ regarded as a perfectly conducting surface and put six equal charges on it. They will certainly distribute like the points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$, or a rotation of this pattern. Now add one more

Figure 1: Picture of a skeleton obtained by a fluid dynamic experiment corresponding to Laplacian growth. Taken from [1]

charge. The seven charges want to be as far from each other as possible, according to the Coulomb law, but what does this mean for the charge configuration? How shall the charges cope with the fact that there is no canonical way to distribute seven points on a sphere?

For some pictures and further information in a more general situation, see [3].

2) Newton discovered, by a nontrivial calculation, that the exterior gravitational field of a homogeneous ball is identical to that of a point mass at the center. This point mass can be thought of as a potential theoretic skeleton of the ball. Does e.g. a homogeneous cube have a similar skeleton? And would the orbit of the moon be different if the earth had the shape of a cube instead of that of a ball?

Potential theoretic skeletons relate to several topics in mathematics and physics, e.g., asymptotic distributions of zeros of orthogonal polynomials in the complex plane (cf. [4]), growth problems in fluid mechanics (Laplacian growth), crystallization, aggregation of particles under Brownian motion (DLA).

References

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