To Professor Harold Shapiro in his 75th Anniversary

ANALYSIS ON LEMNISCATES AND HAMBURGER'S MOMENTS

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ABSTRACT. We study the metric and analytic properties of generalized lemniscates $E_t(f) = \{z \in \mathbb{C} : \ln|f(z)| = t\}$ where f is an analytic function. Our method involves the integral averages $W(t) = \int_{E_t(f)} |w(z)|^2 |dz|$, $w(z)$ is a meromorphic function. The present basic result states that the length of the generalized lemniscates as the function of t is just bilateral Laplace transform of a certain positive measure. It follows that $W(t)$ is a positive kernel, i.e. the Hankel matrix $\|W^{(i+j)}(x)\|_{i,j=0}^{\infty}$ is positively definite and the sequence $W^{(k)}(t)$, $k = 0, 1, \ldots$ forms a Hamburger moments sequence. As a consequence, we establish convexity of $\ln|E_t(f)|$ outside of the set of critical values of $\ln|f|$. In particular, in the polynomial case this implies various extensions of some results due to Eremenko, Hayman and Pommerenke concerning one Erdös conjecture. As another application, we develop a method which gives explicit formulae for certain length functions. Some other applications to analysis on lemniscates are also discussed.

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1. INTRODUCTION

We consider a generalized lemniscates

$$\ln|f(z)| = t$$

where $f(z)$ is an arbitrary analytic function provided that the level sets contain compact components (it is the case when $\ker f \neq \emptyset$). As the main result of the present paper we

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regard a discovery of a surprising connection between the problem of level sets length evaluating and the classical Stieltjes moments theory (actually, the Hamburger moments in our context). In fact, we also consider the general case of Riemannian surfaces foliating by harmonic level sets.

We show that the average of any meromorphic function over the generalized lemniscates as the function of t is just bilateral Laplace transform of a certain positive measure. To establish this fact we develop a special technique of differentiating over the harmonic level sets and embed the initial problem to an appropriate formalism of M-systems on vector bundles.

Our starting point is the polynomial lemniscates.

1.1. **Erdös conjecture.** Let $P(z) = z^n + a_1 z^{n-1} + \ldots + a_n$, $n \geq 2$ be an arbitrary monic polynomial. In [EHP] Erdös, Herzog and Piranian have considered series of problems concentrated around the metric properties of t-lemniscates

$$E_t(P) = \{z \in \mathbb{C} : |P(z)| = e^t\}$$

(see Figure 1). One of them is

ERDÖS CONJECTURE [EHP, Problem 12],[Erd] *For fixed degree n of P, is the length of the lemniscate $|P(z)| = 1$ greatest in the case where $P(z) = z^n - 1$? Is the length at least 2π, if $E_0(P)$ is connected?*

Concerning the first part of Erdös conjecture, which is still unsolved in present, Thomas Erdelyi cites that "this problem seems almost impossible to settle" [Erl]. We can add that even a more particular problem of evaluating $|E_t(P)|$ seems to us to be very hard.

It is worth to say that there is principal difference between two following problems: evaluating of length of a fixed lemniscate (the well-known example is the Bernoulli lemniscate) and the problem of studying the length as a function of t.

The first problem essentially answers for rather algebraic than analytic aspects and is closely related to the algebraic geometry and number theory. We also have to mention that it very related to the periods theory [KZ].

On the other hand, the second problem can only be viewed at first sight as a problem from the special functions area. Nevertheless, we think that this problem (to be almost undeveloped at present) should also lead to important intermediate results. Here we are making an attempt to demonstrate this point by showing that the *length function problem* is in tight connection with such areas as Hamburger's moments, (log)convex functions, hypergeometric functions and potential theory.

One of the sources for this problem in the proper sense is due to one paper by Piranian [Pr]. At the same time, the main obstacle for the exploration is almost missing of any explicit formulae for $|E_t(P)|$ (excluding the trivial polynomials $P(z) = (z-a)^n$). The only completely studied ones are the rose-type polynomials $Q_n \equiv z^n - 1$ [Bu] (see also [El]). It is interesting to note that their role in the metric polynomial theory is similar to that one of the Köbe function in the theory of univalent functions.

The second part of Erdös conjecture is related to the lower estimate of the length $|E_0(P)|$ for so-called K-polynomials, i.e. polynomials with connected lemniscate $E_0(P)$. This problem was solved in affirmative by Pommerenke in [Pm1] by establishing that

$$\min_{P \in K_n} |E_0(P)| = |E_0((z-a)^n)| = 2\pi. \tag{1}$$

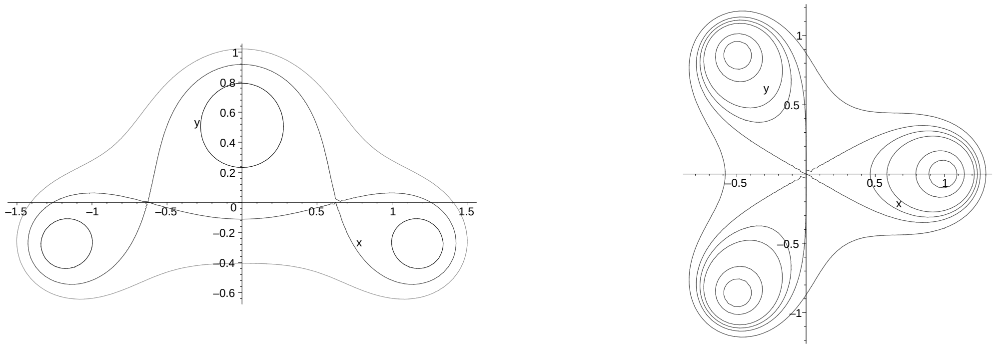

FIGURE 1. Typical lemniscates for D-polynomials: $P(z) = z^3 - \frac{3}{4^{2/3}} z + \frac{i\sqrt{3}}{2}$ and $Q_3(z) = z^3 - 1$

To state our main results we remind some recent results and related standard terminology. We say that $T \in \mathbb{R}$ is the *critical value* (of $\ln |P|$) if there is a root $\zeta \in \ker P'$ of the first derivative such that $|P(\zeta)| = e^T$; the corresponding lemniscate $E_T(P)$ is then called to be *singular*. On the other hand, it is a simple consequence of Morse theory that $E_t(P)$ is a finite collection of smooth closed curves provided that t is a regular value. We also refer to $(\alpha; \beta)$ as a *regular interval* if it contains no critical values of $\ln |P|$.

1.2. **Basic definitions.** By $|E_t(P)| \equiv \mathcal{H}^1(E_t(P))$ we denote the Hausdorff one-dimensional measure of $E_t(P)$ which is a continuous function of all $t \in \mathbb{R}$ (see also section 4). Sometimes, instead of $E_t(P)$ we use the special notation for the non-reduced level set

$$\mathcal{E}_\tau(P) = \{z : |P(z)| = \tau\},$$

whence $\mathcal{E}_\tau(P) = E_t(P)$, $t = \ln \tau$. Given a domain D we write $E_{t,D}(P) = E_t(P) \cap D$.

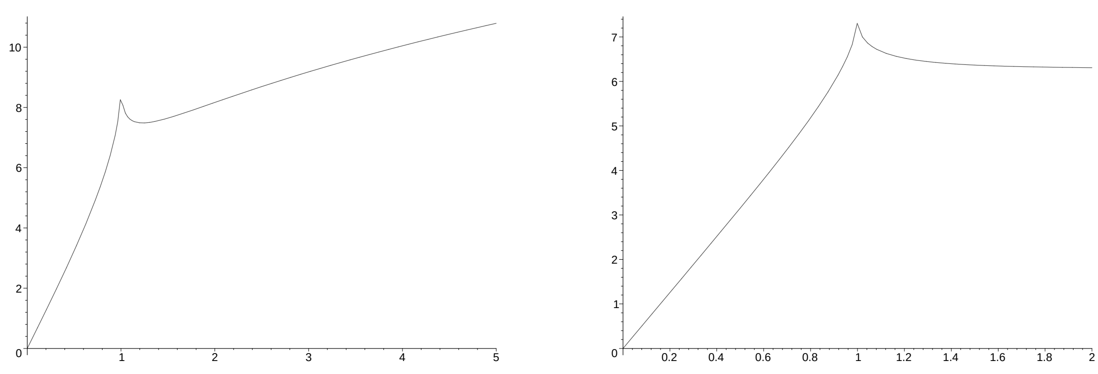

FIGURE 2. The length functions $|\mathcal{E}_\tau(\varphi)|$ for $\varphi = Q_3 = z^3 - 1$ and $\varphi = \sin z$

The actual breakthrough in the Erdös conjecture has been done by Alexander Eremenko and Walter Hayman in recent paper [EH]. They proved that an *extremal* polynomial $P^*(z)$ does exist for an arbitrary degree n and also the following linear estimate

$$c_n \equiv \max_{\deg P = n} |E_0(P)| \le An,$$

with $A = 9,173\dots$ holds. One can easily show that the conjectural value is $|E_0(Q_n)| = 2n + O(1)$. Other upper estimates were due to Pommerenke [Pm2]: $c_n \le 74n^2$ and P. Borwein [Br]: $c_n \le 8\pi e n$.

We call $P(z)$ a *D-polynomial* if the equality $|P(\zeta_k)| = 1$ holds for all critical points $P'(\zeta_k) = 0$, i.e. $\ker P' \subset E_0(P)$. One can easily prove that monic D-polynomials are contained in algebraic surface of constant discriminant

$$|\operatorname{Dis} P| = n^n.$$

By using quasiconformal mappings methods in [EH] the following important property of the extremal polynomials has been established: *the lemniscate $E_0(P^*)$ is always connected (in other words, P^* is always a K-polynomial); moreover, there exists an extremal D-polynomial for any degree n.* Then the fact that $Q_2(z-a)$, $a \in \mathbb{C}$, are just only D-polynomials for $n = 2$ implies the affirmative answer in Erdös conjecture for this degree.

In this paper we refine the last property by showing that the lemniscate $E_0(P^*)$ of every extremal polynomial P^* is always singular (see Corollary 1.3).

As it was mentioned above, we propose here an alternative approach to study the metrical properties of lemniscates. Roughly speaking, instead of variation of polynomials with fixed level set, we consider the variations of the level heights for a fixed polynomial. As we see below, in a sense both view points are dual to each other.

The another key idea we exploited is that the lemniscates of polynomials are level sets of harmonic functions. This fact is well known and frequently used, but nobody seems to have studied the *arc-length behavior* by using the fact. In this connection we have to mention the Green function method which is described in Marden's monograph [Mr]. On the other hand the notions of lemniscate and ribbon domains we are using below are borrowed from theory of zero-mean curvature surfaces [Mk1], [MkT1], [MkT2] (their authors use a terminology of relativistic bands instead of a ribbon).

The way how we are using the arc-length function can be explained as follows. Let us introduce the *indicator* of a polynomial $P(z)$, $\deg P = n$, by

$$\Phi_P(t) = \ln |E_t(P)| - \frac{t}{n}.$$

Then the geometrical sense of $\Phi_P(t)$ as the reduced length of a level set is very clear. Really, the homothetic polynomial

$$P_t(z) \equiv e^{-t} P(z e^{\frac{t}{n}}) = z^n + a_1 e^{-\frac{t}{n}} z^{n-1} + \ldots + a_{n-1} e^{-\frac{t(n-1)}{n}} z + a_n e^{-t} \tag{2}$$

is also monic with coefficients continuously depending on $t \in (-\infty; +\infty]$, $P_0(z) \equiv P(z)$ and, what is more

$$E_x(P_t) = \{z \in \mathbb{C} : |P_t(z)| = e^x\} = \{z \in \mathbb{C} : |P(z e^{\frac{t}{n}})| = e^{t+x}\} = e^{-\frac{t}{n}} E_{t+x}(P). \tag{3}$$

Hence, we have the following equivariant with respect to action of (2) identity

$$\Phi_{P_t}(x) = \ln |E_x(P_t)| - \frac{x}{n} = \ln(e^{-\frac{t}{n}} |E_{t+x}(P)|) - \frac{x}{n} = \Phi_P(x+t). \tag{4}$$

Thus, we can regard the reduced arc-length function, or indicator, as a slice of variation of the lemniscates lengths through a specific direction. In particular,

Proposition 1.1. *Let if $P^*(z)$ be an extremal polynomial of degree n then $t = 0$ is a global maximum point of $\Phi_{P^*}(t)$.*

Proof. Clearly, it follows from (4) that

$$\Phi_{P^*}(t) = \Phi_{P_t^*}(0) = |E_0(P_t^*)| \le |E_0(P^*)| = \Phi_{P^*}(0).$$

□

Unfortunately, it implicitly follows from Theorem 1.2 below that the extreme points of both the length function and the indicator function are never smooth because these functions turn out to have cusp points at the corresponding critical values T_k (see Figure 2). That is why the direct classical variational methods do not work properly near of T_k.

1.3. **Main results.** An important feature of our approach is the fact that the most of results we establish for the level sets of arbitrary holomorphic functions. The only requirement we need in that case is compactness of the corresponding level sets (or at least their components).

Let $f(z)$ be an arbitrary analytic function with $\ker f \neq \emptyset$ (e.g. f is a monic polynomial or $\sin z$). To avoid extra explanations we assume here (nevertheless, see the exact definitions in the text below) that $f(z)$ is an entire analytic function. We refer to f as to a *foliating* function and by $E_f(t)$ we denote the lemniscate set $\ln|f(z)| = t$ provided that it contains compact components. Let $w_0(z)$ be a meromorphic function such that all its poles are contained in the singular t-lemniscates only.

We consider the following first-order differential operator

$$G_f[w_0] \equiv w_1 = 2g(z)w'(z) + g'(z)w(z), \qquad w_k = G_f^k[w_0],$$

where $g(z) = f(z)/f'(z)$ and $w_k(z)$ are the iterations of w_0 under G_f. Let

$$s_0(t) = \int\limits_{E_t(f)} |w_0(z)|^2 \, |dz|.$$

Then we show (Theorem 4.3) that the Hankel matrix

$$\begin{pmatrix} s_0(t) & s_0'(t) & s_0'(t) & \dots \\ s_0'(t) & s_0''(t) & s_0'''(t) & \dots \\ s_0''(t) & s_0'''(t) & s_0^{(4)}(t) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

of kth derivatives of the initial term $s_0(t)$ has the form of the Gram matrix in a suitable Hilbert space:

$$s_k(t) = s_0^{(k)}(t) = \int\limits_{E_t(f)} \operatorname{Re} \overline{w}_{k-j} w_j \, |dz|, \qquad 0 \le j \le k, \tag{5}$$

which implies that the sequence

$$s_k(t) \equiv \int\limits_{E_t(f)} |w_k(z)|^2 \, |dz|$$

forms a Stieltjes moments consequence for all regular t (this corresponds to Hamburger problem in classical analysis). This observation, in turns, leads us to the Bernstein-Widder bilateral representation

Theorem 1.1. *Let f and w_0 be as above. Then the following representation holds*

$$\int\limits_{E_t(f)} |w_0(z)|^2 \, |dz| = \int\limits_{-\infty}^{+\infty} e^{xt} \, d\sigma_{f,w_0}(x), \tag{6}$$

where σ_{f,w_0} is a non-negative measure on $(-\infty; +\infty)$ and t ranges in an arbitrary regular interval.

We use this fact implicitly in our evaluating method of the length functions for various functions f in Section 5.

We notice that another equivalent to (6) property is that function $s_0(t)$ is an *exponentially convex* (e.c.) function, i.e. the associated with $s_0(t)$ stationary kernel $s_0(x + y)$ is positive. The last property in turn means that the Hankel matrix (see (14))

$$\left[s_0 \left(\frac{t_i + t_j}{2} \right) \right]_{i,j=1}^{m}$$

is positively defined for all $m \geq 1$ and any t_i in the regular interval $(\alpha; \beta)$.

This class of e.c. functions $s_0(t)$ was firstly studied by Bernstein [Ber] and Widder [Wd] in connection with the so-called completely (or absolutely) monotonic analytic functions. We should mention a deep penetration of the both classes of functions into complex analysis, inequalities analysis, special functions, probability theory, radial-function interpolation, harmonic analysis on semigroups (Schoenberg theory [BCR]), combinatoric analysis and economics (see recent survey [AB] for further discussion and references).

Another useful consequence of representation (6) is the theorem due to Bernstein [Ber], saying that function $s_0(t)$ admits an analytic continuation in the complex plane $\mathbb{C}$ except of finite number of lines orthogonal critical values T_k.

In the most interesting for us particular case when the foliating function $f(z)$ is just a monic polynomial $P(z)$ and $w_0(z) \equiv 1$.

Corollary 1.1. *In every regular interval $\mathfrak{I} = (\alpha; \beta)$ the following representation holds*

$$\text{length}(E_t(P)) \equiv |E_t(P)| = \int\limits_{-\infty}^{+\infty} e^{xt} \, d\sigma_P(x),$$

with a positive measure σ_P. In particular, the length function $|E_t(P)|$ is exponentially convex.

We are indebted to Serguei Shimorin for suggesting the another method of proving of Corollary 1.1 (see Theorem 6.1). One of the benefit of this kind of representation is that it shows the explicit form of measure σ_P. It turns out that σ_P is supported at a enumerate set.

Another consequence of preceding results is that $\ln |E_t(P)|$ (as well as the indicator function $\Phi_P(t)$) is *convex* on every regular interval $(\alpha; \beta)$. More precisely,

Theorem 1.2. *The function $\Phi_P(t)$ is continuous for all $t \in \mathbb{R}$. Moreover, if the polynomial $P(z)$ is non-trivial (i.e. is different from $(z - a)^n$) then*

A) *$\Phi_P(t)$ is strongly convex on each regular interval $\mathfrak{I}$;*

B) *the following asymptotic holds*

$$\lim_{t\to+\infty} \Phi_P(t) = \ln 2\pi. \tag{7}$$

The proof of this assertion is given in Section 4 and essentially based on the general result (Theorem 3.1 below) concerning the differentiating of smooth functions over harmonic level sets on arbitrary Riemannian manifold.

An immediate consequence of Theorem 1.2 is the mentioned above lower estimate (1) due to Pommerenke.

Corollary 1.2. *Let P be a monic K-polynomial, i.e. its lemniscate $E_0(P)$ is connected. Then*

$$|E_0(P)| \geq 2\pi,$$

with equality only in the case $P(z) = (z-a)^n$.

Proof. Really, in the case $P(z) = (z-a)^n$ one can readily check that $|E_0(P)| = 2\pi$. So, we can assume that $P(z)$ is different from $(z-a)^n$. Then by according to the definition of K-polynomial its critical values are non-positive: $t_k \leq 0$, and by virtue of Theorem 1.2 it follows that $\Phi_P(t)$ is strongly convex on the semi-axe $[0;+\infty)$. Moreover, by (7) the function $\Phi_P(t)$ is bounded on $[0;+\infty)$ and as a consequence of convexity $\Phi_P(t)$ is actually strongly decreasing. Because of $\Phi_P(0) = \ln |E_0(P)|$ we have $\Phi_P(0) > \ln 2\pi$, or $|E_0(P)| > 2\pi$ which completes the proof. $\square$

By the mentioned above Eremenko-Hayman theorem, we know that for all integer $n \geq 2$ extremal polynomials P^*, $\deg P = n$, do exist.

Corollary 1.3. *If $P^*(z)$ is an extremal polynomial of nth degree then the lemniscate $E_0(P^*)$ is singular, i.e. it contains at least one critical point:*

$$\ker P^{*\prime} \cap E_0(P^*) \neq \varnothing.$$

Proof. By Proposition 1.1, $t = 0$ is an absolute maximum of the indicator function $\Phi_{P^*}(t)$ and it follows that it can not be a regular value of $\ln |P(z)|$ because of strong convexity of $\Phi_{P^*}(t)$ in a neighborhood of the regular values. $\square$

Some problems being initially posed by Piranian in [Pr] and dealing with the monotonicity and convexity of the length function $|\mathcal{E}_\tau(Q_n)|$ in the rose-case $Q_n(z) = z^n - 1$ have been studied in papers [Bu], [El]. In section 5 we obtain formula for $|\mathcal{E}_\tau(Q_n)|$ (originally due to Butler [Bu]) as an easy consequence of (5) by reducing to certain hypergeometric differential equation. We point out that most of properties which have been established for polynomials $Q_n(z)$ in [Bu] could be essentially extended for the general case by using our results.

We also mention that property (6) allows us to interpret the indicator $\Phi_P(t)$ as the *Bernstein function.* The last notion plays an important role in mathematical finance and study of the self-similar Markov processes (we refer for details to [BF], [BD]).

Now we return to the case of arbitrary analytic foliating function f. For instance, we consider examples of the level sets: $|\tanh z| = e^t$ or $|\sin z| = e^t$, which are compact in a neighborhood of each zero (see Figure 3).

Then another benefit of formulae (5) is that they can be involved to a linear differential equation on the corresponding length function $H(t)$ provided the foliating function f satisfies certain additional properties.

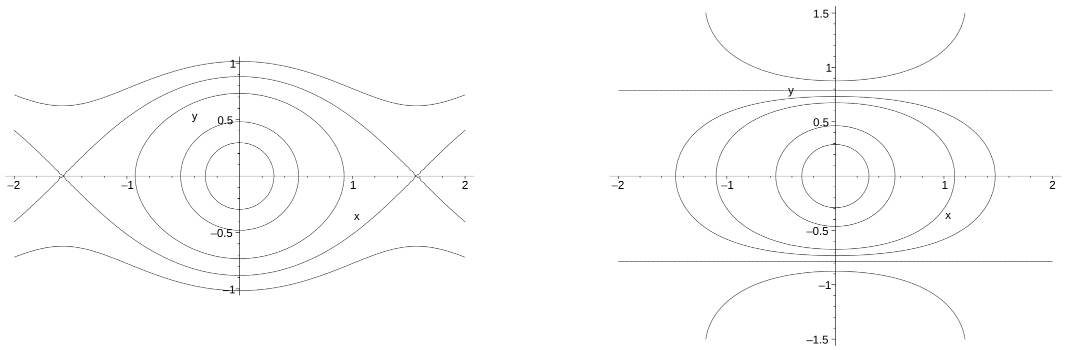

FIGURE 3. Generalized lemniscates: $|\sin z| = e^t$ and $|\tanh z| = e^t$

For example, using this observation we obtain an explicit form of the length function for $f = \sin z$

$$|\mathcal{E}_{\tau,2\pi}(\sin z)| = 2\pi \begin{cases} \tau \quad {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \tau^4\right), & \tau \in [0,1]; \\ \tau^{-2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1/\tau^4\right), & \tau \in [1, +\infty) \end{cases} \tag{8}$$

(see Section 5). Here $\mathcal{E}_{\tau,2\pi}(\sin z)$ denotes the level set $|\sin z| = \tau$ modulo 2π (we emphasize that for $\tau > 1$ the entire level set is already non-compact). The second string of the glue-function in (8) follows from our results on the ribbon domains. Really, in the ribbon case we allow the level sets to have non-empty boundary by compensation of the corresponding Neumann condition. It turns out that for the periodic functions the same formulae as for the pure lemniscate case hold.

We show in Section 5 that the previous results are valid for a whole family of generalized lemniscates. Moreover, three previous functions: Q_n, $\sin z$ and $\tanh z$ are D-functions in our terminology, i.e. they are solutions of the following equation

$$\varphi' = (1 - \varphi^{\nu})^{\frac{k+1}{\nu}}. \tag{9}$$

It is worth to say here that the previous solutions are analogues of D-polynomials because its critical values have the same magnitude: $|\varphi(z_k)| = 1$, $z_k \in \ker f$.

The paper is organized as follows. Because the most results which we establish below can be extended in rather general situation we only indicate our main method, postponing the other details beyond of context of the present paper. In Section 2 we introduce the auxiliary formalism of general λ-moments systems which are the key ingredient in our arguments. This notion is a bridge between our technical assertions concerning the harmonic level set analysis on Riemannian manifolds (which is discussed in Section 3) and the generalized lemniscates. In Section 4 we prove the main results for generalized lemniscates and polynomial ones as well. Therein we establish the exponential convexity property of the arc length function. In Section 5 we demonstrate some applications and obtain explicit formulae for the arc length function in special case (9).

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2. Preliminaries: moments systems

2.1. M-systems. We introduce here an auxiliary abstract notion of M-system and its modifications which make further analysis more clear and gives a deeper explanation of how our method is working. The simplest case which is described below concerns trivial bundles, but it can be considerably extended on the general vector bundles as well.

Let $\mathfrak{I} \subset \mathbb{R}$ be an open interval and $(\mathfrak{V}, \mathfrak{I}, \pi)$ be a trivial smooth vector bundle over $\mathfrak{I}$ with total space $\mathfrak{V}$ and projection map π. We shall denote the bundle by the same letter $\mathfrak{V}$ and assume that each fibre vector space $\pi^{-1}(t)$ is equipped with a smooth inner (scalar) product $\langle u; v\rangle_{\mathfrak{V}}$ where u, $v \in \mathrm{Sec}(\mathfrak{V})$ are smooth sections of $\mathfrak{V}$. It is clear that $\langle u; v\rangle_{\mathfrak{V}}$ is a smooth function of t.

Definition 2.1. Let $G : \mathfrak{V} \to \mathfrak{V}$ be a smooth linear morphism (here it means that $\pi(G(u)) = \pi(u)$). We say that $(\mathfrak{V}, G, D, \mathfrak{I})$ is an *M-system* (actually, a *moment* system) if the first derivative operator $D = \frac{d}{dt}$ agrees to G, i.e. for all pairs of sections $u, v \in \mathrm{Sec}(\mathfrak{V})$

$$2D\langle u; v\rangle_{\mathfrak{V}} = \langle Gu; v\rangle_{\mathfrak{V}} + \langle u; Gv\rangle_{\mathfrak{V}} \tag{10}$$

and

$$D^2\, \|u\|^2_{\mathfrak{V}} = \|Gu\|^2_{\mathfrak{V}}. \tag{11}$$

Remark 2.1. The first identity can be interpreted as the covariant gradient property. Really, the operator ∇ defined as $2D$ on $C^\infty(\mathbb{R})$ and as G on $\mathfrak{V}$ does satisfy the formal Leibnitz rule with respect to the inner product

$$\nabla\langle u; v\rangle_{\mathfrak{V}} = \langle \nabla u; v\rangle_{\mathfrak{V}} + \langle u; \nabla v\rangle_{\mathfrak{V}}.$$

As for the second identity (11), it is more specific but it will be shown in the proof of Theorem 2.1 below that this property is equivalent to

$$\langle u; G^2u\rangle_{\mathfrak{V}} = \langle Gu; Gu\rangle_{\mathfrak{V}}, \qquad \forall u \in \mathrm{Sec}(\mathfrak{V})$$

provided (10) holds. Nevertheless, we use axiom (11) because it arises more naturally in our consequent analysis.

We also use the following brief notations for the kth iteration of a section u:

$$u_k = G^k(u) = \underbrace{G \circ \ldots \circ G}_{k}(u),$$

with an obvious agreement that $u = u_0$. So, relations (10) and (11) can be rewritten as the following ones

$$2D\langle u_0; v_0\rangle_{\mathfrak{V}} = \langle u_0; v_1\rangle_{\mathfrak{V}} + \langle v_0; u_1\rangle_{\mathfrak{V}} \tag{12}$$

and, moreover,

$$D^2\|u_0\|^2_{\mathfrak{V}} = \|u_1\|^2_{\mathfrak{V}}. \tag{13}$$

Further, we need the following well-known definitions (see [Ak, Definition 2.6.3]). Given an interval $I \subset \mathbb{R}$, a sequence (s_k), $k = 0, 1, \ldots$ is said to be *strictly I-positive* if for any polynomial $P(z) = a_0 + a_1 z + \ldots + a_m z^m$ which is positive on I we have

$$a_0 s_0 + a_1 s_1 + \ldots + a_m s_m > 0.$$

We omit the word "strictly" if the inequality sign is only semi-defined.

In what follows, by $\mathrm{Han}[s_0, s_1, \ldots, s_{2n}]$ we denote a Hankel matrix of the form

$$\mathrm{Han}[s_0, s_1, \ldots, s_{2n}] = \begin{pmatrix} s_0 & s_1 & s_2 & \ldots \\ s_1 & s_2 & s_3 & \ldots \\ s_2 & s_3 & s_4 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{14}$$

Because the most interesting for the sequel is the case $I = \mathbb{R}$ we formulate here the well-known corresponding characteristic property due to H. Hamburger.

Hamburger Theorem, [H]**.** *A sequence (s_k) is strictly $\mathbb{R}$-positive if and only if the following equivalent assertions hold:*

(A) *all the Hankel forms $\sum_{i,j=0}^{m} s_{i+j} x_i x_j$ are strictly positive;*
(B) *there exists decreasing function $\sigma(\xi)$ on $\mathbb{R}$ with infinitely many growth points such that the Stieltjes integrals produce sequence (s_k):*

$$\int_{-\infty}^{+\infty} \xi^k \, d\sigma(\xi) = s_k, \qquad k = 0, 1, \ldots \tag{15}$$

Now we are ready to formulate the main result of this section.

Theorem 2.1. *Let $(\mathfrak{V}, G, D, \mathfrak{I})$ be an M-system and $w_0 \in \mathrm{Sec}(\mathfrak{V})$ be an arbitrary smooth section. Then*

$$\langle w_j; w_{n-j} \rangle_{\mathfrak{V}} = \langle w_0; w_n \rangle_{\mathfrak{V}} = D^n \|w_0\|^2_{\mathfrak{V}}, \qquad t \in \mathfrak{I}, \quad 0 \le j \le n. \tag{16}$$

Moreover, the sequence

$$s_{k,w_0}(t) = D^k \|w_0\|^2_{\mathfrak{V}}, \qquad k = 0, 1, \ldots$$

is an $\mathbb{R}$-positive sequence for all admissible $t \in \mathfrak{I}$. Moreover, it is strictly positive if and only if the system $w_0, w_1, \ldots$ is not linear dependent.

Proof. First we notice that by (12) we have

$$D\langle w_0; w_0 \rangle_{\mathfrak{V}} = \langle w_0; w_1 \rangle_{\mathfrak{V}} \tag{17}$$

which coincides with (16) for $n = 1$. Moreover, from (17) and (12) we find

$$2D^2 \langle w_0; w_0 \rangle = 2D\langle w_0; w_1 \rangle_{\mathfrak{V}} = \langle w_1; w_1 \rangle_{\mathfrak{V}} + \langle w_0; w_2 \rangle_{\mathfrak{V}}, \tag{18}$$

whence taking into account (13) we obtain

$$\langle w_1; w_1 \rangle_{\mathfrak{V}} = \langle w_0; w_2 \rangle_{\mathfrak{V}}.$$

The last relation together with (18) shows that (16) is fulfilled for $n = 2$.

At the remaining part of the proof we apply induction by index n. Namely, we assume that (16) holds for all sections $w_0 \in \mathrm{Sec}(\mathfrak{V})$ and all non-negative integers j such that $0 \le j \le n$ where $n \ge 2$ is a step of induction.

Then applying the induction assumption to $n-1$ we obtain for $u_0 = w_1 \in \mathrm{Sec}(\mathfrak{V})$

$$D^{n-1}\,D^2\|w_0\|^2_{\mathfrak{V}} = D^{n-1}\|w_1\|^2_{\mathfrak{V}} = D^{n-1}\|u_0\|^2_{\mathfrak{V}} = \langle u_0; u_{n-1}\rangle_{\mathfrak{V}} = \langle u_j; u_{n-j-1}\rangle_{\mathfrak{V}},$$

for all $0 \le j \le n-1$. Thus, we have

$$D^{n+1}\|w_0\|^2_{\mathfrak{V}} = \langle w_1; w_n\rangle_{\mathfrak{V}} = \langle w_2; w_{n-1}\rangle_{\mathfrak{V}} = \langle w_{j+1}; w_{n-j}\rangle_{\mathfrak{V}}$$

for the same range of j. This proves required induction conclusion for all $1 \le j \le n$.

On the other hand, applying again the induction assumption to n we can write

$$\langle w_0; w_n\rangle_{\mathfrak{V}} = \langle w_1; w_{n-1}\rangle_{\mathfrak{V}}$$

whence applying $2D$ to both sides of the last relation we arrive at

$$\langle w_1; w_n\rangle_{\mathfrak{V}} + \langle w_0; w_{n+1}\rangle_{\mathfrak{V}} = \langle w_2; w_{n-1}\rangle_{\mathfrak{V}} + \langle w_1; w_n\rangle_{\mathfrak{V}},$$

or $\langle w_0; w_{n+1}\rangle_{\mathfrak{V}} = \langle w_2; w_{n-1}\rangle_{\mathfrak{V}}$ which yields the rest of cases $j=0$ and $j=n+1$. Hence, the required property is proved.

To establish the positivity property we observe that the Hankel matrix $\mathrm{Han}[s_0, s_1, \ldots]$ is just the Gram matrix of sections w_j, and hence, it has to be positively semi-definite. The case when the Gram matrix is non-strictly positively definite corresponds to vanishing of some diagonal determinant $\mathrm{Han}[s_0, s_1, \ldots, s_{2k}]$ for some index k that obviously yields linear dependence of the system $\{w_j\}$. □

Now we can apply the theorem due to S.Bernstein [Ber, §. 15] to conclude the following important property of M-systems.

Corollary 2.1 (Bernstein-Widder representation). *Let $(\mathfrak{V}, G, D, \mathfrak{I})$ be an M-system and w_0 be a smooth section in $\mathrm{Sec}(\mathfrak{V})$. Then $\|w_0\|^2$ is an exponentially convex function and there exists (unique in essential) a positive measure $\sigma_{w_0}(x)$ on $\mathbb{R}$ such that the following representation holds*

$$\|w_0\|^2 = \int_{-\infty}^{+\infty} e^{xt}\, d\sigma_{w_0}(x), \qquad t \in \mathfrak{I}. \tag{19}$$

2.2. **λ-moments systems.** Now we consider more general situation which corresponds in sequel to so-called *ribbon domains.* Let $\{a, b\}$ be *anti-commutative* braces

$$\{a, b\} = -\{b, a\}$$

which are defined for any pair of smooth sections of $\mathfrak{V}$.

Definition 2.2. Given $\lambda \in \mathbb{R}$ we call $(\mathfrak{V}, G, D, \mathfrak{I})$ a λ-*moments system* if

(i) it satisfies (12),

(ii) instead of (13) the modified identity holds

$$D^2\|u_0\|^2_{\mathfrak{V}} = \|u_1\|^2_{\mathfrak{V}} + \lambda\{u_0; u_1\}, \tag{20}$$

(iii) the braces satisfy

$$2D\{u_0; v_0\} = \{Gu; v\} + \{u; Gv\} \equiv \{u_0; v_1\} + \{u_1; v_0\}.$$

One obviously sees that in the case $\lambda = 0$ we have an M-system.

Lemma 2.1. *Given* $u_0,\ v_0 \in \mathrm{Sec}(\mathfrak{V})$ *the following identities hold*

$$\langle u_2; v_0\rangle_{\mathfrak{V}} - 2\langle u_1; v_1\rangle_{\mathfrak{V}} + \langle u_0; v_2\rangle_{\mathfrak{V}} = 2\lambda\Bigl(\{u_0, v_1\} + \{v_0, u_1\}\Bigr). \tag{21}$$

In particular,

$$\langle u_{n+2}; u_n\rangle_{\mathfrak{V}} - \|u_{n+1}\|^2_{\mathfrak{V}} = 2\lambda\{u_n, u_{n+1}\}, \qquad \forall n \geq 0. \tag{22}$$

Proof. By (20) we have

$$\begin{aligned} D^2\|u_0 + v_0\|^2_{\mathfrak{V}} = \|u_1 + v_1\|^2_{\mathfrak{V}} + \lambda\{u_0 + v_0, u_1 + v_1\} = \|u_1\|^2_{\mathfrak{V}} + 2\langle u_1; v_1\rangle_{\mathfrak{V}} + \\ + \|v_1\|^2_{\mathfrak{V}} + \lambda\{u_0, u_1\} + \lambda\{u_0, v_1\} + \lambda\{v_0, u_1\} + \lambda\{v_0, v_1\}. \end{aligned} \tag{23}$$

On the other hand, we can simplify the left side of the last identity by using linearity of D

$$\begin{aligned} D^2\|u_0 + v_0\|^2_{\mathfrak{V}} &= D^2\|u_0\|^2_{\mathfrak{V}} + 2D^2\langle u_0; v_0\rangle_{\mathfrak{V}} + D^2\|v_0\|^2_{\mathfrak{V}} = \\ &= \|u_1\|^2_{\mathfrak{V}} + \lambda\{u_0, u_1\} + 2D^2\langle u_0; v_0\rangle_{\mathfrak{V}} + \|v_1\|^2_{\mathfrak{V}} + \lambda\{v_0, v_1\}, \end{aligned}$$

and, what is more, by (12)

$$2D^2\langle u_0; v_0\rangle_{\mathfrak{V}} = D\left(\langle u_0; v_1\rangle_{\mathfrak{V}} + \langle u_1; v_0\rangle_{\mathfrak{V}}\right) = \frac{1}{2}\left(\langle u_2; v_0\rangle_{\mathfrak{V}} + 2\langle u_1; v_1\rangle_{\mathfrak{V}} + \langle u_0; v_2\rangle_{\mathfrak{V}}\right).$$

After comparing the expressions obtained with (23) we arrive at (21). □

It follows form (22) that unlike the case of M-systems, for $\lambda \neq 0$ does not in general hold coincidence of scalar products in (16) but it is still true some analogue of Theorem 2.1.

Lemma 2.2 (Leibnitz Rule)**.** *Given an arbitrary integer* $n \geq 1$ *the following identity holds*

$$2^n D^n\langle u_0; v_0\rangle = \sum_{k=0}^{n} C_n^k\langle u_k; v_{n-k}\rangle.$$

In particular,

$$2^n D^n\|u_0\|^2_{\mathfrak{V}} = \sum_{k=0}^{n} C_n^k\langle u_k; u_{n-k}\rangle.$$

Proof. For $n = 1$ the desired identity follows from (12) and by induction similar classical Newton binomial theorem, for other $n \geq 2$ (though in our context this lemma rather relates to the Leibnitz rule). □

3. Averages over harmonic level sets

3.1. **Lemniscate domains.** Below we establish the basic technical results. We arrange the arguments in the general case of arbitrary Riemannian manifolds. Methods developed here allow us to consider the whole spectrum of problems concerning the analysis along harmonic level sets. For one of such applications, namely theory of tubes and relativistic bands with zero-mean curvature, we refer to [Mk1], [MkT2]. Some other generalizations can be found in [Tk].

Let M be a p-dimensional Riemannian manifold equipped by the inner scalar product $\langle X; Y\rangle$ and covariant derivative ∇; by $\operatorname{div} X$ we denote the divergence operator generated by ∇. A function $u(x) : M \to \mathbb{R}$ is called harmonic if $\Delta u \equiv \operatorname{div}\nabla u(x) = 0$ and we denote by $E_t(u)$ the level set $\{x \in M : u(x) = t\}$.

Definition 3.1. A triple $(D, u, \mathfrak{I})$ where $D \subset M$ is an open subset with compact closure, $u(x)$ is C^2-smooth function and $\mathfrak{I} = (\alpha; \beta)$ is said to be an *lemniscate domain* if for all t from $(\alpha; \beta)$ the set $E_t(u) \cap D$ is compact in D. The function u used to be called an *foliating* function of the lemniscate domain (cf. with [Mk1]).

Theorem 3.1. *Let $(D, u, \mathfrak{I})$ be a lemniscate domain with harmonic function $u(x)$. Let $h(x)$ be a C^2-smooth function on D and*

$$H(t) = \int\limits_{E_t(u)} h(x)|\nabla u(x)|\, d\mathcal{H}^{p-1}(x), \tag{24}$$

where by $d\mathcal{H}^{p-1}$ we denote $(p-1)$-dimensional Hausdorff measure on $E(t)$.

Then $H(t)$ is a C^2-function in a neighborhood of arbitrary regular value $\tau \in \mathfrak{I}$. Moreover,

$$H'(\tau) = \int\limits_{E_\tau(u)} \frac{\langle \nabla h(x); \nabla u(x)\rangle}{|\nabla u(x)|}\, d\mathcal{H}^{p-1}(x), \tag{25}$$

and

$$H''(\tau) = \int\limits_{E_\tau(u)} \frac{\Delta h(x)}{|\nabla u(x)|}\, d\mathcal{H}^{p-1}(x). \tag{26}$$

Proof. Because of regularity there exists an ε-neighborhood of τ such that all the level sets $E_t(u)$ are embedded submanifolds in M and everywhere along $E_t(u)$ the vector field

$$\nu(x) \equiv \frac{\nabla u(x)}{|\nabla u(x)|} \tag{27}$$

represents the fields of unit normals in $D_t(u) = \{x \in D : u(x) < t\}$ to $E(t)$. Hence

$$\langle \nu(x); \nabla u(x)\rangle = |\nabla u(x)|. \tag{28}$$

Then we claim that for arbitrary C^1-vector field $\mathbf{v}$ defined in a neighborhood of $E_{t_0}(P)$ and

$$F(\tau) \equiv \int\limits_{E_t(u)} \langle \mathbf{v}; \nu\rangle\, d\mathcal{H}^{p-1}(x)$$

the following auxiliary formula holds at every regular value τ:

$$F'(\tau) = \int\limits_{E_\tau(u)} \frac{\operatorname{div} \mathbf{v}}{|\nabla u|}\, d\mathcal{H}^{p-1}(x). \tag{29}$$

Really, let $t \in (\tau - \varepsilon; \tau + \varepsilon)$ different from τ be chosen arbitrary. Then by virtue of (28) and harmonicity of $u(x)$ we have by Stokes' formula

$$F(t) - F(\tau) = \int\limits_{E_t(u) - E_\tau(u)} \langle \mathbf{v}; \nu\rangle\, d\mathcal{H}^{p-1}(x) = \int\limits_{\partial D(\tau,t)} \langle \mathbf{v}; \nu\rangle\, d\mathcal{H}^{p-1}(x) = \int\limits_{D(\tau,t)} \operatorname{div} \mathbf{v}\, dx, \tag{30}$$

where $D(\tau, t) = D(t) - D(\tau)$ (here and what follows we consider $D(t)$ and $E_u(t)$ as chains to avoid extra signs if $t < \tau$).

We recall the co-area formula (see [F, § 3.2] and [BZ]) which in our case takes the form

$$\int\limits_{D(\tau,t)} g(x)\,dx = \int\limits_{\tau}^{t} d\xi \int\limits_{E_\xi(u)} \frac{g(x)\,d\mathcal{H}^{p-1}(x)}{|\nabla u(x)|}, \tag{31}$$

where $g(x)$ is an arbitrary Borel function. By applying of (31) to (30) we obtain

$$\frac{F(t)-F(\tau)}{t-\tau} = \frac{1}{t-\tau}\int\limits_{\tau}^{t} d\xi \int\limits_{E_\xi(u)} \frac{\operatorname{div}\mathbf{v}}{|\nabla u|}\,d\mathcal{H}^{p-1}(x). \tag{32}$$

The last limit does exist at every regular point τ (even if u is only locally Lipschitzian in D) by Kronrod-Federer theorem whence by regularity of τ we conclude that the derivative $H'(\tau)$ does exist and (32) yields (29).

Thus, applying (29) to $\mathbf{v} = h\nabla u$ we have from harmonicity of u and (24) that (25) holds.

Now we observe that by using of (27) the relation (25) can be written in the following form

$$H'(\tau) = \int\limits_{E_\tau(u)} \langle \nabla h(x); \nu\rangle\, d\mathcal{H}^{p-1}(x).$$

Again applying (29) to the last relation with $\mathbf{v} = \nabla h(x)$ we obtain

$$H''(\tau) = \int\limits_{E_\tau(u)} \frac{\operatorname{div}\nabla h(x)}{|\nabla u|}\, d\mathcal{H}^{p-1}(x)$$

which implies (26) and the theorem is proved. □

Remark 3.1. Actually, the previous assertion is still true with suitable arrangements if instead of harmonic function we consider harmonic forms. But a more interesting feature of the latter method is that in the holomorphic case we can already evaluate all higher derivatives (see the following section in partial case when $u(x) = \ln|P|$ with P to be a polynomial).

We should also mention that the harmonic lemniscate domains are typical for the minimal tubes in the Euclidean space. This type of surfaces admit a harmonic coordinate function which level sets are compact (the well-known example is catenoid). Moreover, the closely related objects, the so-called maximal tubes in the Minkowski space can be interpreted as closed relativistic strings (we refer to [Mk2] for other details).

3.2. **Ribbon domains.** Another simple but helpful extension of Theorem 3.1 can be treated if we consider a slightly changed notion of lemniscate domains.

Definition 3.2 (cf. [Mk1]). We call a triple $(D, u, \mathfrak{I})$ a *ribbon domain* with respect to a smooth function $u(x)$ and an foliating interval $\mathfrak{I} = (\alpha; \beta)$ if

(i) the open domain $D \Subset M$ has piecewise smooth boundary ∂D;

(ii) at every regular point of $x \in \partial D$ one holds either $(u(x)-\alpha)(u(x)-\beta) = 0$ or $\langle \nabla u; \mu\rangle = 0$ where μ is the unit normal to ∂D in M.

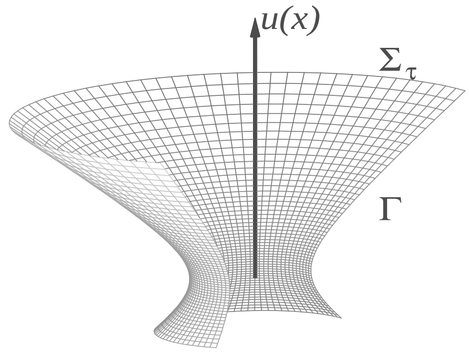

FIGURE 4. A ribbon domain with Σ and Γ components of ∂D (arrow shows the direction of growth of u)

The last definition essentially says that the boundary of a ribbon domain consists of a finite number of smooth surfaces which are either the level sets or gradient-sets of $u(x)$ (in sequel we define them as E and Γ components respectively). We recall that a smooth embedded submanifold Γ is a *gradient-set* of a function u if the gradient field of u is tangent to Γ.

A typical example of a ribbon domain is a precompact domain on a two-dimensional Riemannian surface with boundary consisting of alternating pairs of level and gradient curves of a smooth function (see the general situation of Riemannian surfaces in [Sp]) or *minimal and maximal bands* in [MkT2], [KM].

Theorem 3.2. *Let $(D, u, \mathfrak{I})$ be a ribbon domain with respect to a harmonic function $u(x)$. Let $\Sigma_t(u) = E_t(u) \cap D$ be the corresponding t-level set and $h(x)$ be a C^2-smooth function on D. If*

$$H_D(t) = \int\limits_{\Sigma_t(u)} h(x)|\nabla u(x)|\, d\mathcal{H}^{p-1}(x),$$

then $H_D(t)$ is a C^2-function in a neighborhood of arbitrary regular value $\tau \in \mathfrak{I}$ and

$$H_D'(\tau) = \int\limits_{\Sigma_\tau(u)} \frac{\langle \nabla h(x); \nabla u(x)\rangle}{|\nabla u(x)|}\, d\mathcal{H}^{p-1}(x), \tag{33}$$

$$H_D''(\tau) = \int\limits_{\Sigma_\tau(u)} \frac{\Delta h}{|\nabla u|}\, d\mathcal{H}^{p-1}(x) - \int\limits_{\partial_D \Sigma_\tau(u)} \frac{\langle \nabla h; \mu\rangle}{|\nabla u|} d\mathcal{H}^{p-2}(x). \tag{34}$$

Here $\partial_D \Sigma_\xi(u)$ denotes the relative to D boundary of $\Sigma_\xi(u)$.

Proof. We use the notations of the previous proof. First we establish the modified formula (29) for

$$F(\tau) \equiv \int\limits_{\Sigma_t(u)} \langle \mathbf{v}; \nu \rangle \, d\mathcal{H}^{p-1}(x).$$

Let τ be a regular value from $\mathfrak{I}$ (or an arbitrary value such that the closure $\overline{\Sigma_\tau(u)}$ contains no critical points). Then by the ribbon domain definition the boundary set $D(\tau, t)$ contains two level sets components $\Sigma_t(u)$ and $\Sigma_\tau(u)$, and the gradient set $\Gamma(\tau, t)$ (which one in general may be empty). Let μ be the unit outward normal field along $\partial D(\tau, t)$ and assume for determinacy that $t > \tau$. Then choosing as above the gradient normal field $\nu = \frac{\nabla u}{|\nabla u|}$ along $\Sigma_t(u)$ and $\Sigma_\tau(u)$ we obtain for the chains

$$\partial D(\tau, t) = \Sigma_t(u) - \Sigma_\tau(u) + \Gamma(\tau, t)$$

which implies by Stokes and co-area formulae that

$$\begin{aligned} F(t) - F(\tau) = & \int\limits_{\Sigma_t(u) - \Sigma_\tau(u)} \langle \mathbf{v}; \nu \rangle \, d\mathcal{H}^{p-1}(x) = \int\limits_{\partial D(\tau,t)} \langle \mathbf{v}; \nu \rangle \, d\mathcal{H}^{p-1}(x) - \\ & - \int\limits_{\Gamma(\tau,t)} \langle \mathbf{v}; \mu \rangle \, d\mathcal{H}^{p-1}(x) = \int\limits_{D(\tau,t)} \operatorname{div} \mathbf{v} \, dx - \int\limits_{\Gamma(\tau,t)} \langle \mathbf{v}; \mu \rangle \, d\mathcal{H}^{p-1}(x) = \\ & = \int\limits_{\tau}^{t} d\xi \int\limits_{\Sigma_\xi(u)} \frac{\operatorname{div} \mathbf{v}}{|\nabla u|} \, d\mathcal{H}^{p-1}(x) - \int\limits_{\tau}^{t} d\xi \int\limits_{\partial_D \Sigma_\xi(u)} \frac{\langle \mathbf{v}; \mu \rangle}{|\nabla u|} d\mathcal{H}^{p-2}(x). \end{aligned}$$

Here we use the co-area formula for the last integrals assuming that $\Gamma(\tau, t)$ is an embedded submanifold of M and apply the fact that the gradient ∇u is actually tangent to $\Gamma(\tau, t)$.

Arguing as in previous proof we find that

$$F'(\tau) = \int\limits_{\Sigma_\tau(u)} \frac{\operatorname{div} \mathbf{v}}{|\nabla u|} \, d\mathcal{H}^{p-1}(x) - \int\limits_{\partial_D \Sigma_\tau(u)} \frac{\langle \mathbf{v}; \mu \rangle}{|\nabla u|} d\mathcal{H}^{p-2}(x). \tag{35}$$

Let now consider $\mathbf{v} = h \nabla u$ which yields by harmonicity of u and (35) that

$$H_D'(\tau) = \int\limits_{\Sigma_\tau(u)} \frac{\langle \nabla h; \nabla u \rangle}{|\nabla u|} \, d\mathcal{H}^{p-1}(x) - \int\limits_{\partial_D \Sigma_\tau(u)} h(x) \frac{\langle \nabla u; \mu \rangle}{|\nabla u|} d\mathcal{H}^{p-2}(x).$$

As according to the ribbon domain definition (ii) the last integral vanishes we arrive at (33).

Similarly, substituting $\mathbf{v} = \nabla h$ into (35) we have by virtue of (33) that (34) holds and the theorem is proved. □

4. Proofs of main results

4.1. **Averages of meromorphic functions.** Let $f(z)$ be a holomorphic function such that there exists a bounded open component $D \Subset \mathbb{C}$ of the set

$$D_f(\alpha, \beta) = \{z \in \mathbb{C} : \alpha < \ln |f(z)| < \beta\}.$$

The most typical examples of f are the polynomials with an arbitrary choice of interval $\mathfrak{I} = (\alpha, \beta) \subset \mathbb{R}$.

Actually, a rather large store of others admissible f's does exist. In particular, given a holomorphic function $f(z)$ with non-empty zeroes set there exists a small sufficiently $\varepsilon > 0$ such that $D_f(a;b)$ does contain bounded components, $a < b < \ln \varepsilon$.

In any case, we assume that a triple $(D, \ln|f|, \mathfrak{I})$ forms a lemniscate domain. Moreover, let

$$E_t(f) = \{z \in \mathbb{C} : \ln|f(z)| = t\}$$

and denote by

$$E_{t,D}(f) = E_t(f) \cap D$$

a t-lemniscate of f, $t \in \mathfrak{I}$. This lemniscate is called *singular* if it contains a zero of the derivative $f'(z)$ (or what is the same, a null of the gradient field $\nabla \ln|f|$) and by T we denote the corresponding critical value of $\ln|f|$. A value t is said to be *regular* if it is different from critical ones.

On the other hand, similar constructions lead us to the ribbon domain examples. Nevertheless, we should emphasize that a ribbon domain construction is considerably simpler because we need no special topological restrictions on f in this case. Roughly speaking, a typical ribbon domain can be viewed as an appropriate "tubular neighborhood" of some gradient curve of $u = \ln|f(z)|$. In standard hydrodynamical terminology gradient curves are just *flow lines* of the corresponding harmonic function u and there is a vast mathematical area (e.g. related to the Green functions) which covers these facts.

Let us fix a function $f(z)$ and associate with it the function

$$g(z) = g_f(z) = \frac{f(z)}{f'(z)}.$$

and the following differential operator

$$G_f[w_0](z) \equiv w_1(z) = 2w_0'(z)g(z) + w_0(z)g'(z) = \frac{1}{w_0}(w_0^2 g)', \tag{36}$$

acting on an arbitrary meromorphic in D function $w_0(z)$.

We say that $w_0(z)$ is *admissible* for $\mathcal{D} = (D, \ln|f|, \mathfrak{I})$ if its poles ζ are contained in the singular lemniscates of f and define

$$W(t) = \int\limits_{E_{t,D}(f)} |w_0(z)|^2 \, |dz|.$$

In general, if the poles set of w_0 is non-empty, $W(t)$ can occur to be infinite at critical values T_k of $\ln|f|$.

Theorem 4.1. *Let $\mathcal{D} = (D, \ln|f|, \mathfrak{I})$ be a lemniscate domain and $w_0(z)$ be admissible for $\mathcal{D}$. Then the function $W(t)$ is of C^∞ in a neighborhood of every regular $t \in \mathfrak{I}$ and the following identities hold*

$$\begin{aligned} W^{(2\nu+1)}(t) &= \int\limits_{E_{t,D}(f)} \operatorname{Re} \overline{w_\nu(z)} w_{\nu+1}(z) \, |dz|, \\ W^{(2\nu+2)}(t) &= \int\limits_{E_{t,D}(f)} |w_{\nu+1}(z)|^2 \, |dz|, \end{aligned} \tag{37}$$

where $W^{(\nu)}$ is ν-derivative, $\nu \geq 0$.

Proof. We coordinate the notations of Theorem 3.1 by $E_t(u) = E_{t,D}(f)$ and identify a complex number $z = x+iy$ with the point $(x, y) \in \mathbb{R}^2$ so the the gradient of a real-valued function $h(x, y)$ takes the form $h'_x + ih'_y$. Then by Cauchy-Riemann theorem we have for an arbitrary holomorphic function $F(z)$

$$\nabla \operatorname{Re} F(z) \equiv (\operatorname{Re} F(z))'_x + i(\operatorname{Re} F(z))'_y = \overline{F'(z)}, \tag{38}$$

whence

$$\overline{\nabla u(z)} = \overline{\nabla \operatorname{Re} \ln f(z)} = \frac{f'(z)}{f(z)} = \frac{1}{g(z)}. \tag{39}$$

Because of local character of our evaluations below we can choose without loss of generality the logarithm branch such that $\ln 1 = 0$.

Let now t be an arbitrary regular value. Then by admissibility condition the function $w_0(z)$ is well-defined and regular at a neighborhood of the lemniscate $E_{t,D}(f)$. Applying (39) for $W(t)$ we arrive at

$$W(t) = \int\limits_{E_{t,D}(f)} \frac{|w_0(z)|^2}{|\nabla u(z)|} |\nabla u(z)|\,|dz| = \int\limits_{E_{t,D}(f)} |w_0^2(z)g(z)|\,|\nabla u(z)|\,|dz|.$$

Hence, if we write $h(z) = |w_0^2(z)g(z)|$ the following derivative exists by (25) and

$$W'(t) = \int\limits_{E_{t,D}(f)} \frac{\langle \nabla h(x); \nabla u(x)\rangle}{|\nabla u(x)|}\,|dz|. \tag{40}$$

To evaluate $\nabla h(z)$ we note that

$$\ln h(z) \equiv \operatorname{Re} \ln w_0^2(z)g(z),$$

and it follows from (38)

$$\overline{\nabla h(z)} = h(z)\frac{d}{dz}(\ln w_0^2(z)g(z)) = \frac{|w_0^2 g|}{w_0 g} w_1 = \overline{w_0} w_1 \frac{|g|}{g}. \tag{41}$$

On the other hand, using the standard representation of scalar product in $\mathbb{R}^2 = \mathbb{C}$ by the real part we obtain from (39) and (41)

$$\langle \nabla u; \nabla h\rangle = \operatorname{Re}\left(\overline{\nabla u} \nabla h\right) = \frac{1}{|g|} \operatorname{Re} \overline{w_0} w_1. \tag{42}$$

Substituting (42) and (39) into (40) we get the needed formula

$$W'(t) = \int\limits_{E_{t,D}(f)} \operatorname{Re} \overline{w_0} w_1\,|dz|. \tag{43}$$

To find the second derivative $W''(t)$ we notice that $\ln h(z)$ is a harmonic function as the real part of $\ln(w_0^2 g)$. It follows then

$$0 = \Delta \ln h(z) = \frac{\Delta h(z)}{h(z)} - \frac{|\nabla h(z)|^2}{h^2(z)},$$

and by (41) we arrive at

$$\Delta h(z) = \frac{1}{h(z)} |\nabla h(z)|^2 = \frac{|w_0 w_1|^2}{|w_0^2 g|} = \frac{|w_1|^2}{|g|}.$$

Substituting the found relations in (26) and taking into account that $\nabla u = 1/|g|$ we obtain from (39)

$$W''(t) = \int\limits_{E_{t,D}(f)} |w_1|^2 \, |dz|. \tag{44}$$

Now we observe that, by virtue of its definition, w_1 and the consequent iterations w_ν for $\nu \geq 2$ are meromorphic too. On the other hand, the singularities of w_ν can only be enlarged at the expense of the poles of $g(z)$, i.e. zero set $\ker f'(z)$. It follows that $w_1(z)$ and w_ν will be admissible too.

It is clear now that (37) can be obtained by recurrence of (43) and (44) for higher derivatives and theorem is proved. □

4.2. **Simple ribbon domains.** To state the corresponding assertion for the ribbon domain we recall notation of the corresponding level set $\Sigma_t(f) = E_t(f) \cap D$. Unlike the pure lemniscate case the boundary of $\Sigma_t(f)$ is non-empty in general. Moreover, $\Sigma_t(f)$ can contain more than one component.

Below we restrict ourselves by the case when $\mathcal{D}$ is a *simple* ribbon domain which means that

- D is a simply-connected domain;
- $f(z)$ analytic in a neighborhood of D;
- both $f(z)$ and $f'(z)$ have no zeroes in D
- the Γ-boundary of D (see definition in 3.2) contains two distinct non-empty components, we denote them by $\Gamma_A \cup \Gamma_B$.

One can readily show that more complicated ribbon domains can be splitted into a union of the simple ones. This fact follows from the standard topological properties of the gradient and flow lines of harmonic functions (on a Riemannian surface in general case), see e.g. [Ms]. Some simple domains can admit boundary components Σ_τ being degenerated in a point (so-called, apexes). We refer to the typical situation on the Figure 5.

Taking into account orthogonality of ∇u and the outward normal μ to Γ we can choose a natural orientation of Γ by the normal field μ_Γ outside of points $\zeta_k \in \ker f'$ such that

$$\langle \mu_\Gamma \wedge \nabla u; e_1 \wedge e_2 \rangle > 0 \tag{45}$$

where $e_1 = 1$, $e_2 = i$ is the standard basis of $\mathbb{C}$. By the B-component of Γ we call that component which outward (with respect to D) normal μ agrees with μ_Γ and another one we call the A-component.

Theorem 4.2. *Let $\mathcal{D}$ be a ribbon domain with interval $\mathfrak{I} = (\alpha; \beta)$ and $w_0(z)$ be admissible for $\mathcal{D}$. Then the function*

$$W(t) = \int\limits_{\Sigma_t} |w_0(z)|^2 \, |dz|.$$

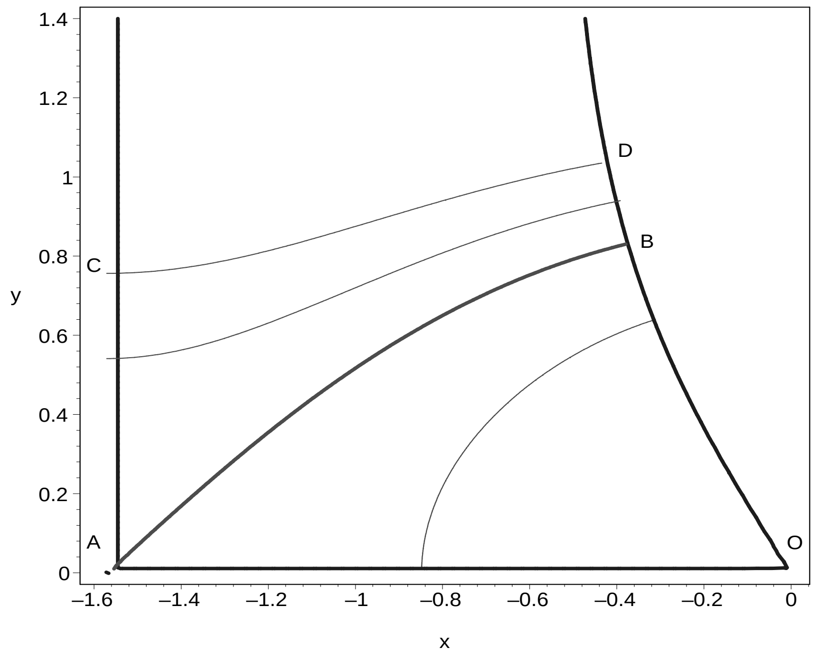

FIGURE 5. Two sin-ribbon simple domains are produced by $f(z) = \sin z$: $ABDC$ and OAB (with degenerate apex O). The Γ components are solid lines: OA, OB for OAB, and AC, BD for $ABDC$. AB is Σ_0-level set which separates two domains.

satisfies

$$
\begin{aligned}
W'(t) &= \int\limits_{\Sigma_t} \operatorname{Re} \overline{w_0(z)} w_1(z)\, |dz|, \\
W''(t) &= \int\limits_{\Sigma_t} |w_1(z)|^2\, |dz| + |g| \operatorname{Im} \overline{w_0} w_1 \Big|_{\partial\Sigma_t(f)}
\end{aligned}
\tag{46}
$$

Proof. The first identity can be immediately obtained by the same way as in the previous proof (see the corresponding property (33) in Theorem 3.2).

To prove the second relation we need only to simplify the last term in (34). We notice that in our notations this integral can be rewritten as

$$
\int\limits_{\partial_D \Sigma_\tau(u)} \frac{\langle \nabla h(x); \mu \rangle}{|\nabla u|} d\mathcal{H}^0(x) = \frac{\langle \nabla h; \mu_\Gamma \rangle}{|\nabla u|}\Big|_{B_t} - \frac{\langle \nabla h; \mu_\Gamma \rangle}{|\nabla u|}\Big|_{A_t}
$$

where $\partial\Sigma_t = B_t - A_t$ is understood as a 0-dimensional chain. By virtue of (45) we have in complex variables

$$
\mu_\Gamma = -i \nabla u
$$

which implies by (39) and (41)

$$
\langle \nabla h; \mu_\Gamma \rangle = \operatorname{Re} \overline{\nabla h}\ \mu_\Gamma = \operatorname{Im} \overline{\nabla h}\ \nabla u = |g| \operatorname{Im} \overline{w_0} w_1
$$

It yields the required identity and the theorem is proved. □

4.3. M**-systems formalism.** To study further properties of $|E_t(f)|$ for both lemniscate and ribbon domains we adopt the obtained results to the formalism of M-systems and λ-moments systems by an appropriate formalism of Hilbert spaces.

At first we remark that the expressions in (37) can be well interpreted if we consider the Hilbert space scale $\mathfrak{H}(t) = \mathfrak{H}^2[E_t(P)]$ of all complex value functions integrable with squared on $E_t(P)$ (actually, we need only the subspace of admissible meromorphic functions) with canonical scalar product

$$\langle f_1; f_2\rangle_{\mathfrak{H}(t)} = \int\limits_{E_t(P)} \mathrm{Re}(\overline{f_1(z)} f_2(z))\,|dz|.$$

In this case $W(t) = \|w_0\|^2_{\mathfrak{H}(t)}$ and by virtue of (37) we have for lemniscate domains

$$W^{(2k)}(t) = \|w_k\|^2_{\mathfrak{H}(t)}, \qquad W^{(2k+1)}(t) = \langle w_k; w_{k+1}\rangle_{\mathfrak{H}(t)}. \tag{47}$$

For any admissible functions v_0 and u_0 let $w_0 = v_0 + u_0$. Then by (47) we have

$$D\|u_0 + v_0\|^2_{\mathfrak{H}(t)} = \langle u_0 + v_0; u_1 + v_1\rangle_{\mathfrak{H}(t)} = \langle u_0; u_1\rangle_{\mathfrak{H}(t)} + \langle v_0; v_1\rangle_{\mathfrak{H}(t)} + \langle u_0; v_1\rangle_{\mathfrak{H}(t)} + \\ + \langle u_1; v_0\rangle_{\mathfrak{H}(t)} = D\|u_0\|^2_{\mathfrak{H}(t)} + D\|v_0\|^2_{\mathfrak{H}(t)} + \langle u_0; v_1\rangle_{\mathfrak{H}(t)} + \langle u_1; v_0\rangle_{\mathfrak{H}(t)}$$

and it follows from linearity of D that

$$D\langle u_0; v_0\rangle_{\mathfrak{H}(t)} = \langle u_0; v_1\rangle_{\mathfrak{H}(t)} + \langle u_1; v_0\rangle_{\mathfrak{H}(t)}.$$

So, we have

Proposition 4.1. *The operator $D = \frac{d}{dt}$ agrees with G_f in the sense of Definition 2.1.*

Moreover, we introduce the anti-commutative braces for two meromorphic functions v and w by

$$\{v, w\}_{\mathfrak{H}(t)} = |g|\,\mathrm{Im}\,\overline{v}w|_{\partial\Sigma_t(f)}\,. \tag{48}$$

Then the second identity in (46) can be rewritten as

$$W''(t) = \langle w_0; w_1\rangle_{\mathfrak{H}(t)} + \{w_0, w_1\}_{\mathfrak{H}(t)}. \tag{49}$$

Lemma 4.1. *The braces (48) satisfy (iii) in Definition 2.2.*

Proof. Let $v(z)$ and $w(z)$ be meromorphic functions and t a regular value of f. At first we notice that $D = \frac{d}{dt}$ is just a restriction of the gradient-like operator $\nabla u/|\nabla u|^2$ on the Γ-components because

$$DF^*(t) = \frac{dF^*(t)}{dt} = \langle \nabla F; \frac{\nabla u}{|\nabla u|^2}\rangle$$

for every smooth function $F(z)$, where $F^*(t) = F(z)\mid_\Gamma$.

Then taking into account (38) we obtain

$$D|g| = |g| D\ln|g| = |g|^3\,\mathrm{Re}\,\frac{g'}{g}\frac{1}{\overline{g}} = |g|\,\mathrm{Re}\,g'.$$

On the other hand, we have (by using usual notations $z = x + iy$)

$$\langle \nabla\,\mathrm{Im}\,\overline{v}w; \nabla u\rangle = u'_x\,\mathrm{Im}(\overline{v'_x}w + \overline{v}w'_x) + u'_y\,\mathrm{Im}(\overline{v'_y}w + \overline{v}w'_y) = \\ = \mathrm{Im}(\overline{v'_x}u'_x + \overline{v'_y}u'_y)w + \mathrm{Im}\,\overline{v}(u'_x w'_x + u'_y w'_y).$$

But for an arbitrary analytic function w one holds

$$w'_x = w'(z), \qquad w'_y = iw'(z)$$

whence we conclude by (39) that

$$\langle \nabla \operatorname{Im} \overline{v} w; \nabla u\rangle = \operatorname{Im} w\overline{v'}(u'_x - iu'_y) + \operatorname{Im} w'\overline{v}(u'_x + iu'_y) = \operatorname{Im} w\overline{v'}\nabla u + \operatorname{Im} w'\overline{v}\overline{\nabla u} =$$

$$= \operatorname{Im}(w\overline{v'} - \overline{w'}v)\nabla u = \operatorname{Im} \frac{w\overline{v'} - \overline{w'}v}{\overline{g}}.$$

Hence, we have

$$D \operatorname{Im} \overline{v} w = |g|^2 \operatorname{Im} \frac{w\overline{v'} - \overline{w'}v}{\overline{g}} = \operatorname{Im} g(w\overline{v'} - \overline{w'}v)$$

and it follows

$$D\{v, w\} = D(|g|) \operatorname{Im} \overline{v} w + |g| D \operatorname{Im} \overline{v} w = |g| \left[\operatorname{Re} g' \operatorname{Im} \overline{v} w + \operatorname{Im} g(w\overline{v'} - \overline{w'}v)\right]. \tag{50}$$

Here and in what follows we omit for convenience of writing the restriction notation $|_{\partial\Sigma(t)}$ (due to linearity of the restriction operator we obviously avoid a confusion). Now we can similar evaluate by the definition of G_f

$$\{v_1, w\} + \{v, w_1\} = |g| \left(\operatorname{Im} \overline{v_1} w + \operatorname{Im} \overline{v} w_1\right) = |g| \operatorname{Im} \left(\overline{v} w_1 - \overline{w} v_1\right) =$$

$$= |g| \operatorname{Im} \left[\overline{v}(2gw' + g'w) - \overline{w}(2gv' + g'v)\right] = |g| \operatorname{Im} \left[2g(\overline{v}w' - \overline{w}v') + g'(w\overline{v} - v\overline{w})\right] =$$

$$= 2|g| \operatorname{Im} g(\overline{v}w' - \overline{w}v') + |g| \operatorname{Re} g' \operatorname{Im} \overline{v} w$$

which completely agrees to (iii) and finishes the proof. □

Thus, denoting by a total bundle space $\mathfrak{V} = \mathfrak{H} = \cup_{t\in\mathfrak{I}}\mathfrak{H}_t$ and letting $D = d/dt$ we obtain by comparing with Theorems 3.1 and 3.2 the following important property

Theorem 4.3. *In our notations, $(\mathfrak{H}, G_f, D, \mathfrak{I})$ forms a λ-moments system. In fact, we have $\lambda = 0$ in the case of lemniscate domains and $\lambda = 1$ for ribbon domains.*

Now, let us denote by A_α the operator

$$A_\alpha(w) = f^\alpha w.$$

Then the following useful in further considerations immediate consequences of the definition hold:

$$G_f \circ A_\alpha - A_\alpha \circ G_f = 2\alpha A_\alpha, \tag{51}$$

and

$$\langle A_\alpha(u); A_\alpha(v)\rangle_{\mathfrak{H}(t)} = e^{2\alpha t}\langle u; v\rangle_{\mathfrak{H}(t)}. \tag{52}$$

In particular, the latter identity shows that multiplication on a power of f up to a homothety is just an isometric operator. This simple but helpful property allows us to establish explicit formulae for $W(t)$ in the next section.

From other benefits of the previous interpretations we especially remark the following consequences

Corollary 4.1. *Let $f(z)$ be an analytic function and $(D, \ln|f|, \mathfrak{I})$ be a lemniscate domain. By $H(t) = |E_t(f)|$ we denote the arc length function. Then the sequence of consequent derivatives $H^{(k)}(t)$ is positive (see Section 2). In particular, $\ln H(t)$ is a convex function.*

Corollary 4.2. *Let $f(z)$ be an analytic function and $(D, \ln|f|, \mathfrak{I})$ be a lemniscate domain. Then given an arbitrary admissible meromorphic function w_0 there exist a unique positive measure σ_f on $\mathbb{R}$ such that the following representation holds*

$$\int_{E_t(f)} |w_0|^2 \, |dz| = \int_{-\infty}^{+\infty} e^{xt} \, d\sigma_{f,w_0}(x), \qquad t \in \mathfrak{I}. \tag{53}$$

Let $\mathcal{D} = (D, \ln|f|, \mathfrak{I})$ be now a simple ribbon domain produced by an R-periodic function $f(z)$, $f(z+R) = f(z)$. Below we use the notations of Γ-components of paragraph 4.2.

Definition 4.1. We call a domain $\mathcal{D}$ *fundamental* if $\Gamma_B \equiv \Gamma_A \mod R$.

Corollary 4.3. *Let D be a fundamental domain corresponding to a periodic function f. Then for any R-periodic meromorphic functions u and v one holds*

$$\{u, v\}_{\mathfrak{H}(t)} \equiv 0. \tag{54}$$

In particular, for an R-periodic function w_0 the function $W(t)$ satisfies all the conclusions of Theorem 4.1

Proof. First, we notice that for every non-empty level arc

$$\Sigma_t = E_t(f) \cap D$$

is a single component curve with the end-points A_t and B_t which are equal modulo R.

To prove this claim we observe that along Σ_t the harmonic function $u(z) = \ln|f(z)|$ is a constant. On the other hand, by virtue of the definition of a simple ribbon domain the function $u(z)$ is strictly monotonic along Γ_A and Γ_B components. Because of $u(A_t) = u(B_t)$ and a point $\zeta \in \Gamma_B$ does exist such that $A_t = \zeta \mod R$ (by Definition 4.1) we conclude that $\zeta = B_t$.

Thus, it follows that the braces $\{\cdot, \cdot\}$ vanish for every pair of two meromorphic R-periodic functions which implies (54). In turn, this means that $(\mathfrak{H}, G_f, D, \mathfrak{I})$ forms in fact a 0-moments system and the required assertion is proved. □

4.4. **Polynomial lemniscates.** Here we study the polynomial case in more details. By $P(z)$ we denote a monic polynomial. Then the closure of the set

$$D = D(t_1, t_2) = \{z \in \mathbb{C} : t_1 < \ln|P(z)| < t_2\}$$

is compact for arbitrary choice of t_j. It follows that $(D, \ln|P|, \mathfrak{I})$ is a lemniscate domain for every interval $\mathfrak{I}$.

Corollary 4.4. *Let $w_0(z)$ be admissible for $P(z)$. Then $\ln W(t)$ is strongly convex on every regular interval provided that $w_0(z)$ is not a solution to*

$$w_0^2(z) = AP^{k-1}(z)P'(z) \tag{55}$$

for some $k \in \mathbb{R}$ and $A \in \mathbb{C}$.

Proof. Let t be a regular value. Then applying (37) we have from Cauchy inequality

$$W''(t)W(t) - W'^2(t) = \int_{E_t(P)} |w_1|^2 \int_{E_t(P)} |w_0|^2 - \left(\int_{E_t(P)} \operatorname{Re} w_0 w_1 \right)^2 \geq$$

$$\geq \int\limits_{E_t(P)} |w_1|^2 \int\limits_{E_t(P)} |w_0|^2 - \left(\int\limits_{E_t(P)} |w_0||w_1| \right)^2 \geq 0.$$

This implies convexity of $\ln W(t)$.

Assume now that the last inequality turns into equality at a regular point t_0. Then it is easy to see that by Cauchy inequality property everywhere on $E_{t_0}(P)$ the identity holds

$$|w_1(z)| = \lambda |w_0(z)|, \qquad \mathrm{Re}(\overline{w_0(z)} w_1(z)) = \pm |w_1(z) w_0(z)| \tag{56}$$

with suitable choice of a real $\lambda \geq 0$.

The case $\lambda = 0$ gives $w_1 = 0$ whence $w_0^2(z) = A/g(z) = AP'(z)/P(z)$ with a certain $A \in \mathbb{C}$.

By virtue of (56) the remaining case $\lambda > 0$ similarly implies that the fraction $w_1/w_0(z)$ is a real constant everywhere along the lemniscate $E_{t_0}(P)$. By the uniqueness theorem for analytic functions the fraction has to be a constant. We denote it by $k \in \mathbb{R}$. Then $w_0(z)$ is an eigenfunction of $G_P[w_0] = (w_0^2 g)'/w_0$ whence after integration we arrive at (55). □

Remark 4.1. If we assume that the degenerate case (55) in Corollary 4.4 occurs then the corresponding function is just an exponent function

$$W(t) = 2\pi n |A| e^{kt}$$

Really, one should only notice that the integral

$$\int\limits_{E_t(P)} |P'(z)|\, |dz| = 2\pi n e^t$$

can be reduced to the flow of harmonic function $u(z) = \ln |P(z)|$ through over the curve $E_t(P)$ and taking into account independence of the flow of a choice of t, we pass to limit as $t \to \infty$.

Proof of Theorem 1.2. Using the previous notations we take a test function to be $w_0(z) \equiv 1$ which produces the lemniscate length $W(t) = H(t)$. Hence, the log-convexity of $H(t)$ immediately follows from Corollary 4.4.

To establish the strong convexity of $\ln H(t)$ we suppose converse and consider an arbitrary regular point t_0 such that $H''(t_0) = 0$. Then by (55) we arrive at

$$AP'(z)P^{k-1}(z) = 1.$$

Simple arguments show that the last relation holds for a polynomial if and only if it has the form $P(z) = (z - z_0)^n$.

Hence, we can decide that $P(z)$ is different from $(z - z_0)^n$ which implies strong convexity of $\ln H(t) = \ln |E_t(P)|$ and what is more, of the function $\Phi_P(t)$, on any regular interval.

Continuity of $|E_t(P)|$ can be established as follows. We observe that by virtue of (3) the following relation holds

$$|E_t(P)| = e^{\frac{t}{n}} |E_0(P_t)| \tag{57}$$

for all $t \in (-\infty; +\infty]$.

On the other hand, we can apply Eremenko-Hayman lemma [EH, Lemma 4] which states that the lemniscate length $E_0(P)$ is a continuous function of its coefficients. Taking into account that the coefficients of P_t (see their explicit form in (2)) are continuous functions of $t \in (-\infty; +\infty]$ we have from (57) the required property.

Finally, it remains to us to prove (7). Again, applying to the Eremenko-Hayman lemma and (3) we notice that $\lim_{t\to+\infty} P_t(z) = z^n$ whence

$$\lim_{t\to+\infty} |E_t(P)|e^{-t/n} = |E_0(z^n)| = 2\pi.$$

which completes the proof of (7). □

In conclusion of this paragraph we mention a useful asymptotic property of the length of $E_t(P)$ as $t \to -\infty$ for polynomials P without any multiple roots. This is the case when P is a D-polynomial. Really, it follows from easy characterization of multiple root z of P as $z \in \ker P \cap \ker P'$. Then in the D-polynomial class all the roots of $P'(z)$ are situated on the same lemniscate $E_0(P)$ which yields $|P(\zeta_k)| = 1 \neq 0$ where $\zeta_k \in \ker P'$.

We regard the following assertion as a math folklore which is repeatedly cited (see, i.e. [Bu]) but have not found any rigorous proof of that fact.

Lemma 4.2. *Let $P(z)$ has no multiple roots. Then*

$$\lim_{t\to-\infty} H(t)e^{-t} = \sum_{j=1}^{n} \frac{2\pi}{|P'(z_k)|}, \tag{58}$$

where $\{z_k\} = \ker P$.

Proof. We denote by T the minimal value among all $\ln|P(\zeta_j)|$ where $\zeta_j \in \ker P'$. Since $P(z)$ has no multiple roots then $|P(\zeta_j)| \neq 0$ and moreover, $T > -\infty$. It follows from Theorem 1.2 that both $\ln H(t)$ and $\ln(H(t)e^{-t})$ are strongly convex on the ray $(-\infty; T]$.

Let

$$M = \max_{|P(z)|\le e^{T-1}} |P''(z)/P'^2(z)|.$$

Then in view of the choice of T the last quantity is well-defined. On the other hand, taking into consideration $w_0(z) \equiv 1$ we find that

$$w_1(z) = \left(\frac{P}{P'}\right)' = 1 - \frac{PP''}{P'^2},$$

whence

$$|w_1(z) - 1| \le |P(z)|, \qquad t \in (-\infty; T-1].$$

Hence, by virtue of (43) (with application to $w_0 \equiv 1$) we arrive at

$$|H'(t) - H(t)| = \left| \int\limits_{E_t(P)} \mathrm{Re}(w_1(z) - 1)\ |dz| \right| \le Me^t H(t),$$

as $t \in (-\infty; T-1]$. It is equivalent to $|(\ln H(t)e^{-t})'| \le Me^t$.

Thus, it follows that the limit of $(\ln H(t)e^{-t})'$ as $t \to -\infty$ does exist and is equal to 0 which implies after convexity of $\ln H(t)e^{-t}$ that

$$\lim_{t\to-\infty} \ln H(t)e^{-t}$$

does exist as well. Now the exact value of the last limit is easy to obtain by taking into account the asymptotic behavior

$$|P(z)| = |z - \zeta_k|\ (|P'(\zeta_k)| + o(1)), \qquad z \to \zeta_k$$

and convexity property of $E_t(P)$ for large negative t's. $\square$

4.5. **Strictly positive functions.** As we have seen before, in the lemniscate case the matrix

$$\mathrm{Han}[W(t), W'(t), \ldots, W^{(2n)}]$$

generated by average of a meromorphic function $w_0(z)$ is actually the Gram matrix of the vector system w_k in $\mathfrak{H}(t)$:

$$s_{i+j}(t) = \langle w_i; w_j \rangle_{\mathfrak{H}(t)}, \quad i, j = 0, 1, \ldots. \tag{59}$$

We say that given a meromorphic function $w_0(z)$ is *strictly positive* if it is admissible for P and the corresponding Gram matrix (59) is strictly positively defined. It is clear, that the last property is equivalent to positivity of the main minors $\|s_{i+j}(t)\|_{i,j=0}^{N}$ for all $N \geq 1$.

Otherwise, to realize the fact what is a *non*-strictly positive function $w_0(z)$ we observe that in this case one can find a not-trivial sequence of reals λ_i such that

$$\|\lambda_0 w_0 + \lambda_1 w_1 + \ldots + \lambda_N w_N\|_{\mathfrak{H}(t)} = 0.$$

By virtue of the uniqueness theorem for analytic functions we have that the last equality being held on $E_t(P)$ implies identity

$$\lambda_0 w_0 + \lambda_1 w_1 + \ldots + \lambda_N w_N = 0 \tag{60}$$

everywhere in general set of definition of w_i (essentially, $\mathbb{C}$ excluding of both poles w_0 and zeroes of P').

It is worth to notice also that in this case the function $W(t) = \|w_0\|^2_{\mathfrak{H}(t)}$ has to satisfy the differential equation with not-trivial set of coefficients

$$\lambda_0 W(t) + \lambda_1 W'(t) + \ldots + \lambda_N W^{(N)}(t) = 0$$

on any regular interval. One easily sees that the inverse statement is also true: if $w_0(z)$ satisfies (60) with $\sum_{j=0}^{j=N} |\lambda_j|^2 \neq 0$ then it is a non-strictly positive function.

Lemma 4.3. *If polynomial $P(z)$ is different from $(z-a)^n$ then the constant function $w_0 \equiv 1$ is necessarily strictly positive.*

Proof. Let $z_1 \in \ker P'(z)$ be chosen in an arbitrary way such that $z_1 \notin \ker P(z)$. By $\nu \geq 1$ we denote its algebraic multiplicity. Since $P(z) \neq (z-a)^n$, by Gauss-Lucas theorem [Mr] such a root does exist. In this case the leading term of Laurent decomposition of $g(z) = P(z)/P'(z)$ in the neighborhood of z_1 has the form

$$[g(z)]_{z_1} = \alpha (z - z_1)^{-\nu}$$

where $\alpha \neq 0$. We claim that

$$[w_k(z)]_{z_1} = A_k \alpha^k (z - z_1)^{-k(\nu+1)}, \qquad A_k \neq 0 \tag{61}$$

for all $k \geq 1$. Really, for $n = 1$

$$[w_1(z)]_{z_1} = [g'(z)]_{z_1} = -\alpha\nu (z - z_1)^{-\nu-1}.$$

Hence, $A_1 = -\nu \neq 0$ and we assume by induction that (61) holds for all k, $1 \leq k \leq n$ for some $n \geq 1$. Then one can readily obtain from (36) that

$$[w_{n+1}(z)]_{z_1} = -A_n \alpha^{n+1} \frac{2n(\nu+1) + \nu}{(z - z_1)^{(n+1)(\nu+1)}}$$

which implies $A_n = (-1)^n \prod_{j=0}^{n-1}(\nu + 2j(\nu+1)) \neq 0$ and the claim is established.

By virtue of decomposition (61) held at a neighborhood of z_1 relation (60) yields that the set $\{\lambda_j\}$ can only be trivial. The last property proves the required statement completely. □

5. Applications

5.1. ***D*-functions.** In this section we consider examples of evaluation of the length function in some special cases. Our main idea is to involve the first derivatives of $H(t)$ into a certain differential equation by using the following simple observation. Here we outline the method only for the second order differential equations though it can be suitably extended for the higher derivatives in the similar way.

First we recall that by the maximum principal for an analytic function $f(z)$ the level set $|f(z)| = \tau$ for some τ contains compact curves if and only if

$$\ker f \neq \emptyset. \tag{62}$$

Then for some sufficiently small $\tau > 0$ the level set $|f(z)| = \tau$ (or $E_{\ln\tau}(f)$ in our standard notations) is a collection of compact Jordan curves enclosing zeroes $z_k \in \ker f$.

Let w_0 be a meromorphic function which satisfies the relation

$$(\alpha w_0 + \beta w_1) f^{\nu} = \gamma w_0 + \delta w_1, \qquad g(z) = \frac{f(z)}{f'(z)}, \tag{63}$$

where we exclude the trivial case by assuming that

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0 \tag{64}$$

and $\nu > 0$. Here we write as above

$$w_1 = G_f[w_0] = 2g w_0' + g' w_0.$$

It turns out that (63) leads us to the simplest case for evaluation of the averages

$$H(t) = \int\limits_{E_t'(f)} |w_0|^2 \, |dz|,$$

where $E_t'(f)$ is an arbitrary finite union of curves in $E_t(f)$.

Really, in this case everywhere on $E_t'(f)$ the following identity holds

$$e^{2\nu t} |\alpha w_0(z) + \beta w_1(z)|^2 = |\gamma w_0(z) + \delta w_1(z)|^2$$

that after simplification and integration over $E_t'(f)$ arises to the differential equation of the following kind

$$(\beta^2 e^{2\nu t} - \delta^2) H''(t) + 2(\alpha\beta e^{2\nu t} - \gamma\delta) H'(t) + (\alpha^2 e^{2\nu t} - \gamma^2) H(t) = 0. \tag{65}$$

We consider in more details the main case $w_0 = 1$ with $H(t) = |E_t'(f)|$. Then (63) can be rewritten as

$$f^{\nu} = \frac{\gamma + \delta g'(z)}{\alpha + \beta g'(z)}.$$

Moreover, by (62) we can find a zero $z_1 \in \ker f$. Then $f(z)$ is well-defined in a neighborhood of z_1 and $g'(z_1) = 1$. We have from (63)

$$\gamma + \delta = 0,$$

whence $\gamma = -\delta$, moreover it follows from (64) that $\delta \neq 0$. After changing of notations $a = \alpha/\delta$ and $b = \beta/\delta$ we arrive at

$$g'(z) = \frac{af^\nu + 1}{1 - bf^\nu}.$$

Taking into account that

$$d(\ln(f/f')) = \frac{dg}{g} = \frac{af^\nu + 1}{1 - bf^\nu} \cdot \frac{df}{f} = \frac{af^\nu + 1}{1 - bf^\nu} \cdot \frac{df^\nu}{\nu f^\nu}$$

and after integration and choosing an appropriate constant $\varphi(z) = cf(z)$ we obtain

$$\varphi' = C(1 - \varphi^\nu)^{\frac{k+1}{\nu}}, \tag{66}$$

where $k = a/b$.

Definition 5.1. A meromorphic function φ (possibly a multivalued one) is said to be a D-function if it satisfies (66).

In fact one can notice by the definition (66) that for any D-function φ holds

$$|\varphi'(\zeta_k)| = 1, \qquad \forall z_k \in \ker \varphi'$$

which means that the D-functions are analogues of D-polynomials because their critical values have the same magnitudes.

Combining the previous facts we arrive at

Theorem 5.1. *Let $\varphi(z)$ be a solution to (66). Then the length arc function*

$$L(\tau) = \mathcal{H}^1\{z : \ |\varphi(z)| = \tau\}$$

satisfies

$$\tau^2(\tau^{2\nu} - 1)L''(\tau) + [1 + (2k+1)\tau^{2\nu}]\tau L'(\tau) + (k^2\tau^{2\nu} - 1)L(\tau) = 0. \tag{67}$$

Proof. Really, we notice that $\varphi(z)$ gives

$$g'_\varphi(z) = \frac{1 + k\varphi^\nu}{1 - \varphi^\nu},$$

hence $\varphi(z)$ satisfies (63) with

$$\alpha = k, \quad \beta = 1, \quad \gamma = -\delta = -1,$$

whence by virtue of (65) and taking into account that

$$\tau L'(\tau) = H'(\ln \tau), \qquad L''(\tau)\tau^2 + \tau L'(\tau) = H''(\ln \tau)$$

we obtain the required differential equation. □

For further analysis it is helpful to reduce equation (67) to the hypergeometric canonic form. To do this we write

$$L(\tau) = \tau S(\tau^{2\nu}), \qquad x = \tau^{2\nu}.$$

Then after easy transformations we arrive at the equivalent form of (67)

$$x(1-x)S''(x) + \left[1 - x(1 + \frac{k+1}{\nu})\right] S'(x) - \left(\frac{k+1}{2\nu}\right)^2 S(x) = 0. \tag{68}$$

The function φ defined by (66) can be multi-valued in general and $\zeta = \varphi(z)$ is the inverse to the hypergeometric function

$$z = \zeta \; {}_2F_1\left(\frac{1+k}{\nu}, \frac{1}{\nu}; \frac{1+\nu}{\nu}, \zeta^\nu\right).$$

By using the main results in Section 3 one can deduce that the arc length function of multi-valued functions satisfy the similar properties as single-valued ones. We omit here the proof of this assertion.

On the other hand, there is a large store of single-valued functions φ and among them we distinguish the following three types:

Type	$\varphi(z)$	k	ν	$p = \frac{k+1}{\nu}$	C
(i)	$1 - z^n$	$-\frac{1}{n}$	1	$1 - \frac{1}{n}$	$-n$
(ii)	$\sin z$	0	2	$\frac{1}{2}$	1
(iii)	$\tanh z$	1	2	1	1
(iii)	$\dfrac{z-1}{z}$	1	1	1	1

These functions represent the solutions of (66) which have the following behavior:

(i) all the level sets $|\varphi(z)| = \tau$ are collections of compact Jordan curves;

(ii) all the level sets $|\varphi(z)| = \tau$, $\tau < 1$ are collections of compact Jordan curves (while for $\tau > 1$ the level sets are non-compact), $\ker \varphi' \neq \emptyset$;

(iii) all the level sets $|\varphi(z)| = \tau$, $\tau \in \mathbb{R}^+ \setminus \{1\}$, are collections of compact Jordan curves, moreover $\ker \varphi' = \emptyset$ (the length function is unbounded near $\tau = 1$).

Really, we observe that the first derivative of $\varphi(z)$ vanishes at ζ if and only if

$$\varphi^\nu(\zeta) = 1,$$

which corresponds to $\tau = |\varphi(\zeta)| = 1$. We call such a point ζ the critical point of φ (this completely agrees to our terminology in the beginning of the paper).

By the Morse theory this fact implies that all closed components of $|\varphi(z)| = \tau$ are compact for all $0 \leq \tau < 1$. Moreover, by its definition the function $\varphi(z)$ has critical points if and only if the integral

$$\int \frac{d\xi}{(1-\xi^\nu)^{\frac{k+1}{\nu}}}$$

converges at $\xi^\nu = 1$. It in turns is equivalent to $\frac{k+1}{\nu} < 1$.

In the case $\frac{k+1}{\nu} \geq 1$ the function φ does not admit critical points at all, hence the components of $|\varphi(z)| = \tau$ are compact curves for $\tau \geq 0$.

5.2. **Explicit formulae.** Let us now assume that $\mathcal{E}_\tau$ be an arbitrary compact component of $E_{\ln \tau}(\varphi)$ (such that it forms a continuous family of curves for $0 \leq \tau < 1$) surrounding zeroes z_j of φ, $j = 1, \ldots, N$. Then using the formulae for the general solution to (68) in interval $[0; 1)$ we obtain for some λ and μ in $\mathbb{R}$ (see [HTF, item 2.3.1])

$$|\mathcal{E}_\tau| = \tau S(\tau^{2\nu}) = \tau \left({}_2F_1(p, p; 1; \tau^{2\nu})\, \lambda + {}_2F_1(p, p; 2p; 1 - \tau^{2\nu})\, \mu \right),$$

here we write for brevity

$$p = \frac{k+1}{2\nu}.$$

Arguing as in Lemma 4.2 we easily see that the length $|\mathcal{E}_\tau|$ has linear growth at $\tau = +0$ (because zeroes of φ are never the critical points). Thus the multiplier

$${}_2F_1(p, p; 1; \tau^{2\nu})\, \lambda + {}_2F_1(p, p; 2p; 1 - \tau^{2\nu})\, \mu \tag{69}$$

should be bounded at $\tau = +0$. Now applying the Euler formula for hypergeometric function [HTF, item 2.1.3])

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

we see that the first term in (69) is bounded on $[0; 1]$ while the second one has unbounded behavior as $\tau \to +0$. Hence, we have $\mu = 0$ and

$$|\mathcal{E}_\tau| = \lambda \tau\, {}_2F_1(p, p; 1; \tau^{2\nu}).$$

Finally, the exact form of λ can be found by using of asymptotic behavior (see a remark above) as $\tau \to +0$

$$|\mathcal{E}_\tau| = 2\pi\tau \sum_{j=1}^{N} \frac{1}{|\varphi'(z_j)|}.$$

We find that $|\varphi'(z_j)| = |C|$ and hence, we arrive at

Corollary 5.1. *Let φ be an arbitrary analytic function satisfying (66). Then in our notations we have*

$$|\mathcal{E}_\tau| = \frac{2\pi N \tau}{|C|}\, {}_2F_1(p, p; 1; \tau^{2\nu}), \qquad 0 \leq \tau < 1,$$

where N is the number of components of $\mathcal{E}_\tau$.

In the case (iii) we can extend (70) for $\tau > 1$. Really, we notice first that $S(\tau)$ should be bounded at a neighborhood of $\tau = 1$. Then arguing as above we conclude by virtue of the Kummer transformation [HTF, item 2.3.1] that $x^{-p}{}_2F_1(p, p; 1; 1/x)$ is a unique bounded (in a neighborhood of $x = 1$) solution to hypergeometric equation on the ray $[1, +\infty)$. It follows that arc length can be represented as continuous glue-function

$$|\mathcal{E}_\tau| = \tau S(\tau^{2\nu}) = \mu \begin{cases} \tau \;\; {}_2F_1(p, p; 1; \tau^{2\nu}), & \tau \in [0, 1]; \\ \tau^{-k}\, {}_2F_1(p, p; 1; 1/\tau^{2\nu}), & \tau \in [1, +\infty) \end{cases}$$

for appropriate constant μ. Taking into account again the behavior of $|\mathcal{E}_\tau|$ at $\tau = +0$ we obtain

Corollary 5.2. *Let φ be an analytic function satisfying (66) of (i)-type. Then*

$$|\mathcal{E}_\tau| = \frac{2\pi N}{|C|} \begin{cases} \tau \;\; {}_2F_1(p, p; 1; \tau^{2\nu}), & \tau \in [0, 1]; \\ \tau^{-k}\, {}_2F_1(p, p; 1; 1/\tau^{2\nu}), & \tau \in [1, +\infty) \end{cases}, \tag{70}$$

where N is the number of components of $\mathcal{E}_\tau$.

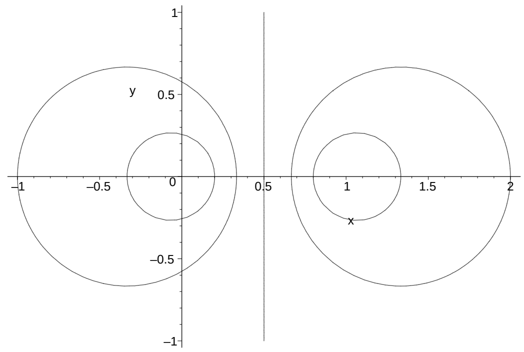

FIGURE 6. Circles family $|z-1| = \tau|z|$.

We demonstrate it by some explicit formulae:

$$|E_\tau(z^n-1)| = 2\pi \begin{cases} \tau \quad {}_2F_1(\frac{n-1}{2n}, \frac{n-1}{2n}; 1; \tau^2), & \tau \in [0,1]; \\ \tau^{1/n}\, {}_2F_1(\frac{n-1}{2n}, \frac{n-1}{2n}; 1; 1/\tau^2), & \tau \in [1,+\infty) \end{cases}, \tag{71}$$

Further we consider the level sets of $\sin z$ and $\tanh z$. Because of their periodicity we choose $\mathcal{E}_\tau$ as an arbitrary closed component. Then for all $\tau \in [0,1]$

$$|\mathcal{E}_\tau(\sin z)| = 2\pi\tau\, {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; \tau^2\right), \tag{72}$$

and

$$|\mathcal{E}_\tau(\tanh z)| = 2\pi\tau\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \tau^4\right), \qquad |\mathcal{E}_{1/\tau}(\tanh z)| = |\mathcal{E}_\tau(\tanh z)|.$$

In fact, formula (72) is still true if we extend it by the general low (71) for $p = 1/4$. Really, it is sufficient to notice that by Corollary 4.3 the ribbon domain $\mathcal{D}$ with $D = [0;\pi/2] \times [0;+\infty)$ and $f(z) = \sin z$ can be obviously regarded as union of two simple ribbon fundamental domains with separate $\mathcal{E}_1$-level arc. Thus, we obtain (8).

Finally, we consider the simplest for the analysis case when $\varphi(z) = \frac{z-1}{z}$ which satisfies (66) with $k = \nu = C = p = 1$. In this case lemniscates $E_\tau(\varphi)$ are the circles with centers at $z_\tau = 1/(1-\tau^2)$ and of radius $r_\tau = \tau/(1-\tau^2)$ (see Figure 6). The direct computations show

$$|E_\tau(\varphi)| = 2\pi r_\tau = \frac{2\pi\tau}{1-\tau^2}$$

which completely agrees with the hypergeometric expression

$${}_2F_1(1,1;1;x) = \frac{1}{1-x}$$

for $p = 1$ and the general formula (70)

Remark 5.1. Excluding trivial case $P(z) = (z-a)^n$, the explicit formulae in the polynomial case for $|\mathcal{E}_\tau|$ are only established for the rose type polynomials $f(z) = Q(z) = z^n - 1$ [Bu], [El]. In those papers the authors use direct computations and further analysis of two different from topological point of view cases $\tau < 1$ and $\tau > 1$. We emphasize that the main benefit of our method is that it gives the total description in both cases by virtue of reducing to unique differential equation.

We recall also that the length function $|\mathcal{E}_\tau|$ can be continued for analytic function defined in the entire complex plane with one line $\operatorname{Re}\tau = 1$ removed (Bernstein result in [Ber] concerning exponentially convex functions). So, we can interpret the glue-function formula (70) as a consequence of the monodromy action around of the singular point $\tau = 1$. The similar situation but of deeper nature and related to the ramified integrals in higher-dimensional complex spaces is studied by Vassiliev in his recent books [Vs1], [Vs2].

Remark 5.2. It follows from (70) the following symmetry property for the type (i) functions φ (e.g., for rose type polynomials)

$$|\mathcal{E}_{1/\tau}| = \tau^{k-1}|\mathcal{E}_\tau|, \qquad \tau \in \mathbb{R}^+.$$

6. Measure σ_P

6.1. **Representation of σ_P.** Here we establish the exponential convexity of the length function for polynomials by using the another approach which has been suggested to us by Serguei Shimorin. We have to mention that the related arguments were used by Pommerenke in [Pm1, Pm2] but the only stationary case $|P(z)| = 1$ was treated there. Nevertheless, that the below approach can not be extended on the Riemannian manifolds case (as well as on ribbon case in two-dimensional complex plane).

One of the main benefit of this method is that it produces the explicit form of measure $\sigma_P(x)$ and shows, e.g., that this measure is actually discontinuous and supported on the specific discrete set consisting of equidistance points on the real axe. It allows us to establish the completely monotonic character of $e^{-t/n}|E_t(P)|$ in $(T, +\infty)$ where T is equal to the maximal singular value of P. Another helpful consequence is that the explicit form of the measure constructed implies examples of the length functions for certain polynomials and analytic functions (cf. with our constructions in paragraphs 5.1 and 5.2).

We demonstrate the main idea of the mentioned method in the partial case only when the lemniscate is a simple curve (i.e. for $t > T$) and postpone the other considerations for a forthcoming paper. But the reader can see that the arguments we use below can be directly extended as well in the case $t < T_{min}$ where T_{min} is the minimal critical value of P. In the general case the results will be true in an appropriate form, but we need some more delicate arguments dealing with ramified coverings.

Theorem 6.1. *Let P be a monic polynomial of degree n and*

$$e^T = \max_{P'(\zeta_k)=0} |P(\zeta_k)|$$

is the greatest singular value of P. Then for all $t \geq T$ the following representation holds

$$|E_t(P)| = \int_{-\infty}^{+\infty} e^{xt}\, d\sigma_P(x), \tag{73}$$

where $\sigma_P(x)$ is a positive discrete measure supported at $\frac{2}{n}\mathbb{Z}^- + \frac{1}{n}$. Moreover,

$$|E_t(P)| = 2\pi e^{t/n} \left(1 + \sum_{k=2}^{+\infty} |c_{-k}|^2 e^{-2kt/n}\right), \tag{74}$$

where c_k are a sequence of the Laurent's coefficients of $\sqrt[n]{P}$ expanded near infinity and

$$\sum_{k\geq 2} |c_{-k}|^2 e^{-2kT/n} < +\infty. \tag{75}$$

Proof. Let $t > T$ be an arbitrary value. Then $E_t(P)$ is a simple Jordan curve which encloses a simply-connected domain $D_t = \{z \in \mathbb{C} : |P(z)| < e^t\}$. Let $D_t^* = \overline{\mathbb{C}} \setminus \overline{D_t}$ be considered as a simply-connected domain with the infinity included.

Then $P(z)$ maps $D^*(t)$ onto $U_t = \{\zeta \in \overline{\mathbb{C}} : |\zeta| > e^t\}$ in such a way that index of P is just its degree n and the auxiliary mapping $F(z) = P^{1/n}(z)$ is well defined in $D^*(t)$. Moreover, $F(z)$ is univalent in $D^*(t)$ and has no critical points there. It is still true for some neighborhood of $D^*(t)$ since $t > T$. We denote by $\varphi(\zeta)$ the inverse mapping which is clearly also univalent conformal mapping of $U_{t/n}$ (and even of a neighborhood of $U_{t/n}$) onto $D^*(t)$ and

$$\varphi'(\zeta) \neq 0, \qquad \zeta \in \overline{U}_{t/n}. \tag{76}$$

Then $E_t(P)$ is a curve with the natural parametrization

$$E_t(P): \quad z = \varphi(\zeta), \qquad \theta \in e^{t/n}\mathbb{T},$$

where $\mathbb{T}$ is the unit circle. So, we have for the length of $E_t(p)$:

$$|E_t(P)| = \int_{e^{t/n}\mathbb{T}} |\varphi'(\zeta)|\,|d\zeta|. \tag{77}$$

We notice that due to (76) (and (82) below) the square root $\sqrt{\varphi'(\zeta)}$ is a well-defined holomorphic function in $\overline{U}_{t/n}$ (which need not be univalent in $\overline{U}_{t/n}$) and it follows that it can be expanded in the Laurent series

$$\sqrt{\varphi'(\zeta)} = \sum_{k=-\infty}^{+\infty} c_k \zeta^k \tag{78}$$

Thus, by virtue of orthogonality of ζ^k on $e^{t/n}\mathbb{T}$, (77) can be rewritten as the following

$$\begin{aligned} |E_t(P)| &= \int_{e^{t/n}\mathbb{T}} |\sqrt{\varphi'(\zeta)}|^2\,|d\zeta| = \int_{e^{t/n}\mathbb{T}} \sum_{k=-\infty}^{+\infty} |c_k|^2 |\zeta|^{2k}\,|d\zeta| = \\ &= \sum_{k=-\infty}^{+\infty} |c_k|^2 \int_{e^{t/n}\mathbb{T}} |\zeta|^{2k}\,|d\zeta| = 2\pi \sum_{k=-\infty}^{+\infty} |c_k|^2 e^{(2k+1)t/n}. \end{aligned} \tag{79}$$

So, we can define a measure $\sigma_P(x)$ by

$$\int_{-\infty}^{+\infty} h(x)\,d\sigma_P(x) = 2\pi \sum_{k=-\infty}^{+\infty} |c_k|^2 h\left(\frac{2k+1}{n}\right) \tag{80}$$

which actually a singular measure with support at $\frac{2}{n}\mathbb{Z} + \frac{1}{n}$ points. Then (79) becomes

$$|E_t(P)| = \int_{-\infty}^{+\infty} e^{xt}\,d\sigma_P(x).$$

Now we describe the support of the measure in more detail. To do it we consider the representation (78) and observe that $P(z) = \zeta^n$ yields that

$$\varphi'(\zeta) = \frac{nP^{\frac{n-1}{n}}(z)}{P'(z)} = \frac{n\zeta^{n-1}}{R(\zeta)},$$

where $R(\zeta)$ is a well-defined holomorphic function in $U_{t/n}$ such that $P'(z) = R(\zeta)$.

On the other hand, we have in $U_{t/n}$

$$\zeta = z\left(1 + \frac{a_1}{z} + \ldots + \frac{a_n}{z^n}\right)^{1/n} = z + \frac{a_1}{n} + \frac{b_1}{z} + \ldots, \tag{81}$$

whence

$$P'(z) = z^{n-1}\left(n + \frac{(n-1)a_1}{z} + \ldots + \frac{a_{n-1}}{z^{n-1}}\right) = \zeta^{n-1}\frac{n + \frac{(n-1)a_1}{z} + \ldots + \frac{a_{n-1}}{z^{n-1}}}{\left(1 + \frac{a_1}{z} + \ldots + \frac{a_n}{z^n}\right)^{(n-1)/n}}.$$

Taking into account that

$$\left(1 + \frac{a_1}{z} + \ldots + \frac{a_n}{z^n}\right)^{-(n-1)/n} = 1 - \frac{(n-1)a_1}{nz} + \left(\frac{(2n-1)(n-1)a_1}{2n^2} - \frac{(n-1)a_2}{n}\right)\frac{1}{z^2} + \ldots$$

we find

$$R(\zeta) = n\zeta^{n-1}\left[1 + \frac{(n-1)a_1^2 - 2na_2}{2n^2}\frac{1}{z^2} + \ldots\right].$$

Finally, we notice that near infinity the following decomposition follows from (81) $z = \zeta - \frac{a_1}{n} + \frac{B_1}{\zeta} + \ldots$ which implies

$$\varphi'(\zeta) = \frac{1}{1 + \frac{(n-1)a_1^2 - 2na_2}{2n^2}\frac{1}{z^2} + \ldots} = 1 - \frac{(n-1)a_1^2 - 2na_2}{2n^2}\frac{1}{\zeta^2} + \ldots \tag{82}$$

Thus, the required expression is the following

$$\sqrt{\varphi'(\zeta)} = 1 - \frac{(n-1)a_1^2 - 2na_2}{4n^2}\frac{1}{\zeta^2} + \ldots$$

and we have in our notations (see (78))

$$c_0 = 1, \qquad c_{-2} = -\frac{(n-1)a_1^2 - 2na_2}{4n^2}, \ldots \qquad \text{and} \qquad c_k = 0 \qquad \text{for all} \quad k \geq 1.$$

It follows now from (79) that

$$|E_t(P)| = 2\pi e^{t/n}\left(1 + \sum_{k=2}^{+\infty}|c_{-k}|^2 e^{-2kt/n}\right), \qquad t > T$$

and it follows that $|E_t(P)|$ is decreasing in $(T; +\infty)$.

We notice that due to continuity of $|E_t(P)|$ on $[T, +\infty)$ (it follows from mentioned above Eremenko-Hayman result) the right hand side of the last equality is at most $|E_T(P)|$. Because of positivity of the terms and continuity of the length function we have that the previous formula is still true for $t = T$ which proves (75) and completes the proof. □

Corollary 6.1. *In the notations of Theorem 6.1 we have the following lower estimate*

$$|E_t(P)| \geq 2\pi e^{t/n}\left(1 + \left|\frac{(n-1)a_1^2 - 2na_2}{4n^2}\right|^2 e^{-4t/n}\right), \qquad t \geq T. \tag{83}$$

6.2. **Complete monotonicity.** We recall (see [Wd, p. 145]) that the function $f(t)$ is *completely monotonic (or c.m.)* in (a, b) if it has non-negative derivatives of all orders there:

$$(-1)^k f^{(k)}(t) \geq 0. \tag{84}$$

The function $f(t)$ is c.m. in $[a, b)$ if it is continuous there and satisfies (84) in (a, b).

Corollary 6.2. *The function* $|E_t(P)|e^{-t/n}$ *is completely monotonic in* $[T, +\infty)$*. In particular, it can be extended to be analytic in the right half-plane* $\operatorname{Re}\zeta > T$*,* $\zeta = t + is$*.*

Proof. One easily sees that the relation (75) implies the uniform convergence of series (74) in $[T, +\infty)$ as well as their successive derivatives in any ray $[T + \varepsilon, +\infty)$, $\varepsilon > 0$.

The analytic property is a sequence of Bernstein theorem on c.m. functions [Ber] (see also [Wd] and [Ak, Chap. V, § 5]). But in our case it can be established directly by the following natural continuation

$$F(t + is) = 2\pi e^{(t+is)/n} \left(1 + \sum_{k=2}^{+\infty} |c_{-k}|^2 e^{-2k(t+is)/n}\right), \tag{85}$$

where $t > T$ and $s \in \mathbb{R}$. Then it follows from (75) that $F(\zeta)$ is an analytic function in $\operatorname{Re}\zeta > T$ and

$$|E_t(P)| = F(t), \qquad t \in \mathbb{R}.$$

□

A interesting feature of (85) is that the function $F(t + is)$ is $2\pi n$*-periodic*, $n = \deg P$. Thus, $\lambda(z) = F(n \ln z)$ is an single-valued analytic function defined in $|z| > e^T$ and such that

$$\lambda(z) = 2\pi z \left(1 + \sum_{k=2}^{+\infty} \frac{|c_{-k}|^2}{z^{2k}}\right).$$

The last formula shows that $\lambda(z)$ is an *odd* function and due to the previous remarks

$$\lambda(e^{nt}) = |E_t(P)| \tag{86}$$

is the length function lemniscate which is continuous at $t = T + 0$. Moreover, it follows from (75) that $\lambda(z)$ can be continuously extended onto the whole circle $e^T\mathbb{T}$ and what is more,

$$|\lambda(z)| \leq \lambda(|z|), \qquad |z| \geq e^T$$

with equality in positive real points $z > e^T$ only.

Example 6.1. Let $P(z) = z^n - 1$, $n \geq 2$, be a rose-type polynomial. Then in the previous notations we have $P(z) = z^n - 1 = \zeta^n$ and everywhere in $|\zeta| > 1$

$$\sqrt{\varphi'(\zeta)} = \sqrt{\frac{n\zeta^{n-1}}{nz^{n-1}}} = \left(1 + \frac{1}{\zeta^n}\right)^{-(n-1)/2n} = 1 - \frac{p}{1!}\frac{1}{\zeta^n} + \frac{p(p-1)}{2!}\frac{1}{\zeta^{2n}} - \dots$$

Thus, we obtain for λ-function:

$$\lambda(z) = 2\pi z \left(1 + \frac{p^2}{1!}\frac{1}{z^{2n}1!} + \frac{p^2(p-1)^2}{2!}\frac{1}{\zeta^{4n}2!} + \dots + \frac{(p)_k^2}{k!}\frac{1}{\zeta^{2kn}k!} + \dots\right)$$

where

$$(p)_k = p(p-1)\cdots(p-k+1) = \frac{\Gamma(p+k)}{\Gamma(p)}$$

is Pochhammer's symbol.

The expression in the braces can be easily recognized as the hypergeometric function property:

$$\lambda(\zeta) = 2\pi\zeta_2 F_1(p, p; 1, \frac{1}{\zeta^{2n}}), \qquad p = \frac{n-1}{2n}$$

and it follows that for the length function (see (86)) we have

$$|\mathcal{E}_\tau^{1/n}| = \lambda(\tau^{1/n}) = 2\pi\tau^{1/n}{}_2F_1(p, p; 1, \frac{1}{\tau^2}), \qquad \tau \geq 1,$$

where $\mathcal{E}_\tau(P) = E_{\ln \tau}(P)$ (cf. these formulas with those in the next section and (71)).

Remark 6.1. It is clear that the previous example can be directly extended on the class of D-functions (see 5.1).

Remark 6.2. The similar arguments imply that in the case $t < T_{min}$ we obtain absolute monotonicity property $|E_t(P)|$ (i.e. $|E_t(P)|$ and its succeeding derivatives are increasing functions in $(-\infty; T_{min}]$).

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